

Yet more problems for 501

Problems tagged by an asterisk may be especially challenging.

33. Assume that $0 \leq a_{nm} < \infty$ for every pair of positive integers (n, m) . Show that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{nm}.$$

(∞ is allowed as a value of the sum.)

34(a). If μ_1, \dots, μ_n are measures on the measurable space (S, \mathcal{F}) , and a_1, \dots, a_n are positive real numbers then $\mu = \sum_1^n a_i \mu_i$ is also a measure on (S, \mathcal{F}) .

(b). If (μ_1, μ_2, \dots) is an infinite sequence of measures on the measurable space (S, \mathcal{F}) , and (a_1, a_2, \dots) is a sequence positive real numbers then $\mu = \sum_1^\infty a_i \mu_i$ is also a measure on (S, \mathcal{F}) .

35. (Inclusion-exclusion principle) For a measure μ on (S, \mathcal{F}) , and sets $A, B \in \mathcal{F}$, $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$.

36. For a sequence of sets $\{E_n\}$, recall

$$\limsup_n E_n \triangleq \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_i \quad \liminf_n E_n \triangleq \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} E_i.$$

(a). If μ is a measure on (S, \mathcal{F}) and the sets E_n are all measurable, then

$$\mu \left(\limsup_n E_n \right) = \limsup_n \mu(E_n) \quad \text{and} \quad \mu \left(\liminf_n E_n \right) = \liminf_n \mu(E_n).$$

(b). (Borel-Cantelli lemma; easy half) If $\sum_1^\infty \mu(E_n) < \infty$, then $\mu(\limsup_n E_n) = 0$.

37(a). Every countable set in \mathbb{R}^d has Lebesgue measure zero.

(b). Let m_F be the Lebesgue Stieltjes measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ associated to a continuous, increasing function F . Show that $m_F(A) = 0$ if A is countable.

38. Let F be the standard Cantor function on $[0, 1]$. Consider the Lebesgue-Stieltjes measure m_F on $[0, 1]$. Let C be the standard Cantor set in $[0, 1]$. From our previous study of the Cantor set we can conclude that $|C| = 0$ and $|C^c| = 1$, where $|A|$ denotes the Lebesgue measure of A and C^c is the complement of C in $[0, 1]$. Show that $m_F(C) = 1$ and $m_F(C^c) = 0$.

39. Let $G(x)$ be the left continuous function $G(x) \triangleq \begin{cases} 0 & \text{if } x \leq 0; \\ x + 1 & \text{if } x > 0. \end{cases}$

Define $\overset{\circ}{\mu}_G$ on the algebra \mathcal{V} of all finite disjoint unions of left half-open intervals $(a, b]$, by the formula:

$$\overset{\circ}{\mu} \left(\bigcup_1^n (a_i, b_i] \right) = \sum_1^n b_i - a_i$$

in which the intervals $(a_i, b_i]$, $1 \leq i \leq n$ are disjoint. Show that $\overset{\circ}{\mu}_G$ is not continuous from below (or from above). (By way of contrast, if $\hat{\mu}_G$ is defined on the algebra of finite disjoint unions of right half open intervals, by $\hat{\mu}(\bigcup_1^n (a_i, b_i)) = \sum_1^n b_i - a_i$, then $\hat{\mu}_G$ will be continuous from below.

40. A finitely additive outer measure is a measure.

41. Let S be uncountable and define $\mu^*(A) = 1$ if A is an uncountable subset of S , and $\mu^*(A) = 0$ otherwise. Show that μ^* is an outer measure and determine what are the μ^* -measurable sets

42(a). Consider the space $[0, 1]^\infty$. For positive integers n , let π_n be the projection of ℓ_2 on its first n coordinates. Let \mathcal{C}_n denote the algebra $\pi_n^{-1}(\mathcal{B}(\mathbb{R}^n))$. and let \mathcal{C} denote the algebra $\bigcap_{n \geq 1} \mathcal{C}_n$. Let \mathcal{C}_∞ be the σ -algebra generated by $\bigcap_{n \geq 1} \mathcal{C}_n$. Show that $\overset{\circ}{\mu}_\infty(\pi_n^{-1}(U)) = m_n(U)$, where $n \geq 1$, U is a Borel set of $[0, 1]^n$, and m_n is Lebesgue measure, defines a finitely additive measure on (ℓ_2, \mathcal{C}) .

(b). Show that $\overset{\circ}{\mu}_\infty$ extends to measure on $([0, 1]^\infty, \mathcal{C}_\infty)$.

43*. Let ℓ_2 denote the space of all sequences $x = (x_1, x_2, \dots)$ of real numbers such that $\sum x_i^2 < \infty$. It can be shown that ℓ_2 , with the metric

$$d(x, y) \triangleq \sqrt{\sum (x_i - y_i)^2}$$

is a complete metric space.

(a). For positive integers n , let π_n be the projection of ℓ_2 on its first n coordinates. Let \mathcal{C}_n denote the algebra $\pi_n^{-1}(\mathcal{B}(\mathbb{R}^n))$. and let \mathcal{C} denote the algebra $\bigcap_{n \geq 1} \mathcal{C}_n$. Let \mathcal{C}_∞ be the σ -algebra generated by $\bigcap_{n \geq 1} \mathcal{C}_n$. What is the relationship between \mathcal{C}_∞ and the Borel σ -algebra of ℓ_2 ?

(b). Show that $\overset{\circ}{\mu}(\pi_n^{-1}(U)) = m_n(U)$, where $n \geq 1$, U is a Borel set of \mathbb{R}^n , and m_n is Lebesgue measure, defines a finitely additive measure on (ℓ_2, \mathcal{C}) . Show that $\overset{\circ}{\mu}$ does not extend to a measure on (ℓ_2, \mathcal{C}) .

44. (Folland, Chapter 1, problem 18.) Let μ be a finitely additive measure on (S, \mathcal{R}) , where \mathcal{R} is an algebra of subsets of S . Define \mathcal{R}_σ to be the collection of countable unions of sets in \mathcal{R} ; let $\mathcal{R}_{\sigma\delta}$ be the collection of countable intersections of sets in \mathcal{R}_σ . Let μ^* be the outer measure induced by μ : for any $G \subseteq S$,

$$\mu^*(G) \triangleq \inf \left\{ \sum_1^\infty \mu(A_i); G \subseteq \bigcup_1^\infty A_i, A_i \in \mathcal{R} \forall i \geq 1 \right\}$$

(a). Given any $G \subseteq S$ and any $\epsilon > 0$, there exists a set A in \mathcal{R}_σ such that $G \subseteq A$ and $\mu^*(A) < \mu^*(G) + \epsilon$.

(b). (Regularity, generalized) Assume that μ is continuous from below. and assume that $\mu^*(G) < \infty$. Then G is μ^* -measurable if and only if there exists a $B \in \mathcal{R}_{\sigma\delta}$ such that $G \subseteq B$ and $\mu^*(B - G) = 0$.

(c). Show that, if μ^* is σ -finite, (b) is true without the assumption that $\mu^*(G) < \infty$.