

TWENTY QUESTIONS FOR THE FINAL

1. (a) Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be normed spaces. Show that the following conditions on a family $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ are logically equivalent: (1) $\{\|T\| : T \in \mathcal{T}\}$ is bounded in \mathbb{R}^+ ; (2) $\{T : T \in \mathcal{T}\}$ is equicontinuous at some point of X ; (3) $\{T : T \in \mathcal{T}\}$ is equicontinuous at $0 \in X$; (4) $\{T : T \in \mathcal{T}\}$ is equicontinuous at every point of X . (This order may not be the most efficient one to prove round-robin implications.)

(b) Let $(X, \|\cdot\|)$ be a Banach space and $\{\Psi_k\}_{k=1}^\infty$ a pointwise convergent sequence of linear functionals (elements of X^*), *i.e.*, such that $\lim_{k \rightarrow \infty} \Psi_k(x)$ exists for each $x \in X$. Show that there is a single bound for all the $\{\|\Psi_k\|\}_{k=1}^\infty$, and that the linear mapping $X \rightarrow \mathbb{K}$ defined by the pointwise limit is continuous on X . {Use **(a)** and the Baire continuity theorem.}

2. Use **1.** to show: if (X, \mathfrak{M}, μ) is a (nonnegative) σ -finite measure space and f is an \mathfrak{M} -measurable scalar-valued function for which $\int_X fh \, d\mu$ (exists and) is finite for every $h \in L^q$, where $1 \leq q \leq \infty$, then $f \in L^p$.

3. Show: if $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, where $1 \leq p \leq \infty$, then the formula $(f * g)(x) = \int_{-\infty}^\infty f(t)g(x-t) \, dt$ defines a function at almost all $x \in \mathbb{R}$, the function belongs to $L^p(\mathbb{R})$, and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$. {Of course Lebesgue measure and Lebesgue-measurable sets are intended. Fubini-Tonelli are essential here. The original post of this suggested the use of **2.** above, but that is not necessary.}

4. Let (X, \mathfrak{M}) be a measurable space, $\mu \geq 0$ with $\mu(X) = 1$ a measure on (X, \mathfrak{M}) , and $\mathfrak{M}_0 \subseteq \mathfrak{M}$ a sub- σ -algebra of \mathfrak{M} . Show that for each $f \in L^1(X, \mathfrak{M}, d\mu)$ there is a unique function (class) $f_0 \in L^1(X, \mathfrak{M}_0, d\mu)$ for which the relation $\int_X fg \, d\mu = \int_X f_0 g \, d\mu$ holds for every \mathfrak{M}_0 -measurable g for which the integrals are finite. This function is called **(a version of the) conditional expectation** of f with respect to \mathfrak{M}_0 and written $E[f|\mathfrak{M}_0]$. {Use Radon-Nikodým on the measure $f \cdot \mu$, considered as acting on (X, \mathfrak{M}_0) .}

5. In the situation of **4.** above, show that $f \mapsto E[f|\mathfrak{M}_0]$ has the following properties:

- (a) $E[\cdot|\mathfrak{M}_0]$ is linear;
- (b) $f \geq 0 \Rightarrow E[f|\mathfrak{M}_0] \geq 0$;
- (c) $E[g \cdot f|\mathfrak{M}_0] = g \cdot E[f|\mathfrak{M}_0]$ for $g \in \mathcal{L}^\infty(X, \mathfrak{M}_0, d\mu)$, and in all cases if both $f, g \geq 0$, f is \mathfrak{M} -measurable and g is \mathfrak{M}_0 -measurable;
- (d) For $1 \leq p \leq \infty$, $f \mapsto E[f|\mathfrak{M}_0]$ sends $L^p(X, \mathfrak{M}, d\mu) \rightarrow L^p(X, \mathfrak{M}_0, d\mu)$ with $\|E[\cdot|\mathfrak{M}_0]\| = 1$;
- (e) For $p = 2$, $L^2(X, \mathfrak{M}_0, d\mu)$ is a norm-closed subspace of $L^2(X, \mathfrak{M}, d\mu)$, and $f \mapsto E[f|\mathfrak{M}_0]$ is the orthogonal projection onto it.

6. Let $\{f_j(x)\}_{j=1}^\infty$ and $\{g_k(x)\}_{k=1}^\infty$ be orthonormal bases for the Hilbert spaces $L^2(\mathbb{R}^n, \text{Lebesgue}, m_n)$ and $L^2(\mathbb{R}^m, \text{Lebesgue}, m_m)$ respectively. Show that the doubly-indexed set $\{f(x) \cdot g(y)\}_{j,k=1}^\infty$ is an orthonormal basis for the Hilbert space $L^2(\mathbb{R}^{n+m}, \text{Lebesgue}, m_{n+m})$.

7. Let $M \subseteq L^2([0, 1], \text{Lebesgue}, m_1)$ be a(n L^2 -norm-)closed linear subspace consisting entirely of continuous functions (*i.e.*, classes each of which has a [necessarily unique] continuous representative). It can be shown (using the closed graph theorem) that there must exist a constant $K \geq 0$ for which $\|f\|_\infty \leq K \|f\|_2$ for all $f \in M$. Assuming this,

(a) Show that for each $x \in [0, 1]$ there exists a function $g_x \in M$ such that $f(x) = \int_0^1 f(t) \overline{g_x(t)} \, dt$ holds for all $f \in M$;

(b) Show that the dimension of M is at most K^2 . {Let $\{f_j\}_{j \in J}$ be an orthonormal set in M , show that $\sum_{j \in F} |f_j(x)|^2 \leq K^2$ for any finite subset $F \subseteq J$, and consider the implications. The corresponding result for $p \neq 2$ is considerably more difficult.}

8. Show that if $f \in \mathcal{L}^p \cap \mathcal{L}^\infty$ for some $p < \infty$ (any measure space will do), then $f \in \mathcal{L}^r$ for all $p \leq r \leq \infty$ and $\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty$.

9. Let (X, \mathfrak{M}, μ) be a measure space with $\mu \geq 0$ and $\mu(X) = 1$. Clearly the unit ball $\{f : f \in L^\infty, \|f\|_\infty \leq 1\}$ is contained in $L^1(X, \mathfrak{M}, \mu)$. Show that it is norm-compact in $L^1(X, \mu, d\mu)$ if and only if μ is purely atomic. {Show that compactness is impossible if μ is purely continuous; then split μ over the decomposition $X = X_c \cup X_a$ and observe that the pieces inherit the compactness condition. The converse is quite easy. **NB:** as announced in class, this will **not** be on the final.}

10. Show that $\ell^\infty(\mathbb{N})$ is not separable. Then show that it is possible to imbed copies of $\ell^\infty(\mathbb{N})$ in $L^\infty(\mathbb{R}^n, \text{Lebesgue}, m_n)$ —by norm-preserving 1-1 isomorphisms of the algebra structure—and deduce that the latter space is not separable. Generalize this to abstract measure spaces to the extent that you can.

11. The gamma integral is defined for $\text{Re}[s] > 0$ by $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$ and the Riemann zeta function for $\text{Re}[s] > 1$ by $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$. Give a rigorous demonstration of the relation

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$$

for $\text{Re}[s] > 1$, using appropriate convergence theorems (assume $s \in \mathbb{R}$ if you wish).

12. (a) Use the addition formula for the cosine to show that the linear space spanned by the functions $\{\cos n\theta\}_{n=0}^\infty$ is an algebra, and deduce (from what double-barrelled theorem?) that it is dense in $\mathcal{C}[0, \pi]$ (in the uniform norm). Thus deduce that if these functions are suitably normalized, they provide an orthonormal basis of $L^2[0, \pi]$ (with the usual L^2 inner product and norm).

(b) It is not difficult to show that for each $n \in \{0\} \cup \mathbb{N}$ there is a polynomial $T_n(x)$ of degree n with the property $T_n(\cos \theta) = \cos(n\theta)$; *assume* this. Then use the substitution $x = \cos \theta$ and freshman calculus on Riemann integrals to show that the $\{T_n(x)\}_{n=0}^\infty$, suitably normalized, form an orthonormal basis of $L^2([-1, 1], \text{Lebesgue}, \frac{dx}{\sqrt{1-x^2}})$.

13. Use complex exponentials (or any means at your disposal) to compute the sum

$$D_N(t) = \sum_{n=-N}^N e^{int}$$

in “closed form.” Thus compute a closed-form formula, as an integral, for the partial sum $\sum_{n=-N}^N \langle f, e^{in\theta} \rangle e^{in\theta}$ of the Fourier series of a function f on $[-\pi, \pi]$ (assumed to be at least integrable, and extended to be periodic on \mathbb{R} to simplify writing the integral in question).

14. Find the Fourier coefficients of the function $f(\theta) = \theta$ on $[-\pi, \pi]$, and use a Hilbert-space theorem to deduce from what you have found the value of $\zeta(2) = \sum_{n=1}^\infty \frac{1}{n^2}$.

15. Let μ be a finite real measure on (X, \mathfrak{M}) . Show that if $\mu(X) = |\mu|(X)$, then one must have $\mu \geq 0$. Does the same implication hold for complex-valued μ (give a proof or a counterexample)?

16. Compute $\lim_{n \rightarrow \infty} \int_0^\infty \left(1 + \frac{t}{n}\right)^{-n} \sin\left(\frac{t}{n}\right) dt$, justifying the calculations (by appeal to suitable convergence theorems, or otherwise).

17. Suppose you know how to construct Cantor sets of arbitrary measure in $[0, 1]$. Exhibit (a) a subset of $[0, 1]$ that is a dense G_δ of measure 0; (b) a subset of $[0, 1]$ that is of first category but measure 1. Can you exhibit a closed nowhere dense subset of $[0, 1]$ that is of measure 1?

18. Assume what you know about the effect of nonsingular linear transformations of \mathbb{R}^n on n -dimensional Lebesgue measure and measurability of subsets of \mathbb{R}^n . (a) Let $N \subseteq \mathbb{R}$ be a set of Lebesgue measure zero. Show that $N \times \mathbb{R} \subseteq \mathbb{R}^2$ is a set of 2-dimensional Lebesgue measure zero. (b) Let $N \subseteq \mathbb{R}$ be a set of Lebesgue measure zero. Show that $\{(x, y) \in \mathbb{R}^2 : x - y \in N\}$ is a set of 2-dimensional Lebesgue measure zero.

19. (a) There are situations in which one encounters disjoint families $\mathfrak{A} = \{E_\alpha\}_{\alpha \in A}$ of measurable subsets of \mathbb{R}^n with $0 < m(E_\alpha) < \infty$ for each $\alpha \in A$ that are **maximal** with respect to those properties, *i.e.*, there is no properly larger disjoint family of measurable sets of finite positive measure that contains \mathfrak{A} . Show (1) that such a family \mathfrak{A} is necessarily countable; (2) that $\mathbb{R}^n = [\bigcup_{\alpha \in A} E_\alpha] \cup N$, where $m(N) = 0$. (Use what you know about Lebesgue-measurable sets.)

(b) Give an example of a(n abstract) measure space (X, \mathfrak{M}, μ) (with $\mu \geq 0$) in which the analogue of (a) above does *not* hold. What additional condition would you put on (X, \mathfrak{M}, μ) to get the analogue of (a)? Show that your condition implies the desired result.

20. Observe that (by a “shift and change of scale”) the set $\{e^{2\pi i n \theta}\}_{n \in \mathbb{Z}}$ is an orthonormal basis of $L^2([-1/2, 1/2], \text{Lebesgue}, m_1)$ with the usual inner product $\langle f, g \rangle = \int_{-1/2}^{1/2} f(t) \overline{g(t)} dt$. In a similar slight variant of the usual definition of the Fourier transform, define $\widehat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i \omega t} dt$ for $f \in L^1(\mathbb{R}, \text{Lebesgue}, m_1)$.

(a) Show that if $f \in L^1(\mathbb{R})$, then the series (indexed by \mathbb{Z}) in $L^1[-1/2, 1/2]$ whose n -th term is the restriction of $t \mapsto f(t-n)$ to $[-1/2, 1/2]$ —usually written as $\sum_{n \in \mathbb{Z}} f(t-n)$, but with the understanding that it is restricted to the interval $[-1/2, 1/2]$ —is (unordered) convergent in $L^1[-1/2, 1/2]$, and its sum has L^1 -norm on the interval at most equal to $\|f\|_1$. Show also that if we denote this sum (only on $[-1/2, 1/2]$!) by $\varphi(t)$, then the Fourier coefficients of φ are related to the Fourier transform of f by the simple $\langle \varphi, e^{2\pi i n \theta} \rangle = \widehat{f}(n)$. {Chop \mathbb{R} up into disjoint translates of $[-1/2, 1/2]$ by elements of \mathbb{Z} .}

(b) Deduce the **Poisson summation formula**: if both f and \widehat{f} fall off sufficiently rapidly at infinity, say $|f(t)| \leq \frac{\text{const.}}{1 + |t|^\alpha}$ and $|\widehat{f}(\omega)| \leq \frac{\text{const.}}{1 + |\omega|^\alpha}$ for some $\alpha > 1$, then

$$\sum_{n \in \mathbb{Z}} f(t-n) = \sum_{k \in \mathbb{Z}} \widehat{f}(k) e^{2\pi i k t} \tag{\$}$$

with the series on both sides unordered convergent uniformly and absolutely on $[-1/2, 1/2]$ (and therefore, by periodicity, absolutely and uniformly on compacta in \mathbb{R}). {(1) Note that if a Fourier series converges uniformly to a continuous function, it must converge to the same function in the L^2 norm. (2) [Added 12/14/99]: K. Dalili has just convinced me that I have tacitly assumed here that f equals the inverse Fourier transform of \widehat{f} , which is true a. e. but not necessarily pointwise. Changing the value of f at the integers can make (\$) false. Hence one must make here the explicit hypothesis that f is continuous.}

(c) Deduce $\sum_{n \in \mathbb{Z}} \frac{1}{(a^2 + n^2)} = \frac{\pi}{a} \coth(\pi a)$ for $a > 0$ from (b) above and the famous contour integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \cos(\alpha x) dx = \pi e^{-|\alpha|}$$

beloved of generations of complex-variables students. Can you deduce the value of $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$?