V. Stability

Throughout most of this chapter we will consider the system

$$x' = f(t, x), \tag{5.1}$$

where $f(t, x)$ is assumed to be continuous on some domain $D \subset \mathbb{R} \times \mathbb{R}^n$. Roughly speaking, a solution $x(t)$ of (5.1) satisfying some initial condition $x(t_0) = x^0$ is stable if all solutions $\tilde{x}(t)$ initiating at some sufficiently nearby point ($\tilde{x}(t_0) = x^1$, with $x^1 \simeq x^0$) approach or remain near to $x(t)$, as $t \to \infty$, in some appropriate sense. Our goal is to develop criteria by which we may verify that particular solutions are or are not stable.

5.1 Linearized stability analysis of constant solutions

We begin with two precise definitions of the idea of stability, due to Lyapunov.

Definition 5.1: Let $x(t)$ be a solution of (5.1) which is defined on some interval (a, ∞). Then $x(t)$ is *stable* if there exists a $t_0 > a$ such that, with $x^0 = x(t_0)$:

(a) There exists a $b > 0$ such that, if $|x^1 - x^0| < b$, then every solution (on a maximal interval) of the IVP

$$\tilde{x}' = f(t, \tilde{x}), \qquad \tilde{x}(t_0) = x^1, \tag{5.2}$$

is defined for all $t \geq t_0$.

(b) For every $\epsilon > 0$ there exists a δ, with $0 < \delta \leq b$, such that if $|x^1 - x^0| < \delta$ then every solution $\tilde{x}(t)$ of the IVP (5.2) satisfies $|\tilde{x}(t) - x(t)| < \epsilon$ for all $t > t_0$.

The solution $x(t)$ is *asymptotically stable* if (a) and (b) hold and if, in addition:

(c) There exists a $\bar{\delta}$, with $0 < \bar{\delta} \leq b$, such that if $|x^1 - x^0| < \bar{\delta}$ then every solution $\tilde{x}(t)$ of the IVP (5.2) satisfies $\lim_{t\to\infty} |\tilde{x}(t) - x(t)| = 0$.

A solution which is not stable is *unstable.*

Remark 5.1: (a) If we wish to emphasize that these stability conditions refer to $t \to \infty$ we may speak of stability or asymptotic stability *at* ∞ or *on the right.* Stability *at* $-\infty$ or *on the left* is defined similarly.

(b) It is an exercise to show that, at least if f is such that solutions of initial value problems are unique, then continuity in initial conditions implies that if conditions (a) and (b), or (a), (b), and (c), hold for some $t_0 > a$ then they hold for all such t_0.

We may immediately restate Theorem 3.10 as a stability result:

Theorem 5.1: *Let A be an $n \times n$ matrix. Then the solution $x(t) \equiv 0$ of the system $x' = Ax$ is asymptotically stable if and only if* $\operatorname{Re}\lambda < 0$ *for all eigenvalues λ of A; this solution is stable if and only if* $\operatorname{Re}\lambda \leq 0$ *for all eigenvalues λ, and every Jordan block in the Jordan form of A for which the eigenvalue λ satisfies* $\operatorname{Re}\lambda = 0$ *is* 1×1.

Note that the terminology here is slightly different from that used for two dimensional linear systems in Example 4.1. In our current terminology, a stable node or stable spiral point is asymptotically stable, and a center (or the origin in the case that A is the zero matrix or that A has eigenvalues 0 and $\lambda < 0$) is stable but not asymptotically stable. All other critical points classified in Example 4.1 are unstable.

We now turn to the general problem (5.1) in the special case in which there exists a constant solution, which by a shift of the x and t variables we may assume to be $x(t) \equiv 0$ and to be defined on $[0, \infty)$. We will consider the case in which the system (5.1) may be approximated by a linear system for points (x, t) with x near 0; specifically, we will always assume that (5.1) may be written in the form

$$x' = A(t)x + h(t, x), \tag{5.3}$$

where $A(t)$ is an $n \times n$ continuous real matrix defined on some interval $I \supset [0, \infty)$, $h(t, x)$ is a $\mathbb{R}^n$-valued function which is defined and continuous on (at least) $D_\rho \equiv \{(t, x) \mid t \in I \text{ and } |x| < \rho\}$ for some $\rho > 0$, and which satisfies $h(t, 0) \equiv 0$. Our goal is to show that in certain cases the stability of the solution $x(t) \equiv 0$ for the system (5.3) is the same as for the approximating linear system $x' = Ax$. We expect this to be true when h is small compared to the linear term $A(t)x$ and will therefore frequently suppose that h satisfies the following condition (C_η) for some $\eta \geq 0$:

CONDITION (C_η) : There exists a $\rho > 0$ such that h satisfies

$$|h(t, x)| \leq \eta |x|, \tag{5.4}$$

for all $(t, x) \in D_\rho$.

We first give a fairly general asymptotic stability theorem for the case of a constant matrix A.

Theorem 5.2: *Suppose that the matrix $A(t)$ in (5.3) is an $n \times n$ constant matrix A for which all eigenvalues $\lambda_1, \ldots, \lambda_n$ have negative real part. Then there exists an $\eta_0 > 0$ (depending on A) such that, if the function $h(t, x)$ satisfies condition (C_η) for some $\eta < \eta_0$, then $x(t) \equiv 0$ is an asymptotically stable solution of the system (5.3).*

Proof: Set

$$\mu \equiv -\sup_{1 \leq i \leq n} (\operatorname{Re} \lambda_i)$$

and choose $\bar{\mu}$ with $0 < \bar{\mu} < \mu$. Since $\operatorname{Re}(\lambda_i + \bar{\mu}) < 0$ for any i, the function $t^k e^{t(\lambda_i + \bar{\mu})}$ is bounded on $[0, \infty)$ for any $k \geq 0$, that is, there is a constant c_{ik} with $t^k e^{t\lambda_i} \leq c_{ik} e^{t\bar{\mu}}$ for $t \geq 0$. Thus if $J = P^{-1}AP$ is the Jordan form of A, then

$$\|e^{tA}\| \leq \|P^{-1}\| \|P\| \|e^{tJ}\| \leq Ce^{-t\bar{\mu}}$$

for some constant C. We will prove that 0 is asymptotically stable if h satisfies (C_η) for some $\eta < \eta_0 \equiv \mu / C$.

Suppose then that (5.4) holds in D_ρ and that $x(t)$ is any solution of (5.3) with $|x(0)| < \rho$. We may treat $u(t) \equiv h(t, x(t))$ in (5.3) as a known function and apply the variation of parameters formulas (3.7) and (3.8) to see that $x(t)$ must satisfy the integral equation

$$x(t) = e^{tA} x(0) + \int_0^t e^{(t-s)A} h(s, x(s))\, ds. \tag{5.5}$$

Hence, as long as $(t, x(t)) \in D_\rho$,

$$e^{t\bar{\mu}}|x(t)| \le C|x(0)| + C\eta \int_0^t e^{s\bar{\mu}}|x(s)|\, ds.$$

Now apply Gronwall's inequality (taking $u(t) = e^{t\bar{\mu}}|x(t)|$ in (2.6)) to find $e^{t\bar{\mu}}|x(t)| \le C|x(0)|e^{tC\eta}$ or

$$|x(t)| \le C|x(0)|e^{-t(\bar{\mu}-C\eta)}. \tag{5.6}$$

We can now verify of the conditions of Definition 5.1 for asymptotic stability. Let $b = \rho/2C$ and suppose that $x(t)$ is a solution of (5.3) with $|x(0)| < b$. Let β be the upper bound of the set of times for which $x(t)$ is defined and lies in D_ρ; if $\beta < \infty$, then (5.6) implies that for $0 \le t < \beta$, $(t, x(t))$ is contained in the compact subset $[0, \beta] \times \{|x| \le \rho/2\}$ of D_ρ, contradicting the extension theorem Theorem 2.19. Thus $\beta = \infty$ and $x(t)$ is defined and satisfies (5.6) for all times $t \ge 0$, verifying both (a) and, with $\bar{\delta} = b$, (c). Finally, given ϵ, assume without loss of generality that $\epsilon < \rho/2$ and let $\delta = \epsilon/C$; then (5.6) implies that if $|x(0)| < \delta$ then $|x(t)| \le \epsilon$ for all $t \ge 0$. ■

We summarize in the following corollary some situations in which Theorem 5.2 immediately implies asymptotic stability of a constant solution. To state one of these conditions we need a piece of standard terminology. If $g(x)$ is an $\mathbb{R}^m$-valued function defined in a neighborhood of 0 and if there exists a nonnegative function $\eta(r)$ defined for $r > 0$, with $\lim_{r\to 0+} \eta(r) = 0$, such that $|g(x)| \le \eta(|x|)|x|^\alpha$ for some $\alpha \in \mathbb{R}$, then we write $|g(x)| = o(|x|^\alpha)$. We will be particularly interested in the case in which $h(t, x) \in \mathbb{R}^n$ is a function defined in some D_ρ, and in which

$$|h(t, x)| \le \eta(|x|)|x|$$

in D_ρ. Then we write $|h| = o(|x|)$ *uniformly in* t. The reader should verify: $|h| = o(|x|)$ uniformly in t iff $h(t, x)$ satisfies condition (C_η) for every $\eta > 0$.

Corollary 5.3: *(a) Suppose that A is an $n \times n$ matrix for which all eigenvalues have negative real part. If $C(t)$ is an $n \times n$ matrix function which is defined and continuous on an interval containing $[0, \infty)$ and for which the norm $\|C(t)\|$ is sufficiently small, uniformly in t, then $x(t) \equiv 0$ is an asymptotically stable solution of*

$$x' = Ax + C(t)x.$$

(b) Suppose that A is as in (a). If $h(t, x)$ is defined in some D_ρ and is $o(|x|)$ uniformly in t, then $x(t) \equiv 0$ is an asymptotically stable solution of

$$x' = Ax + h(t, x).$$

(c) Suppose that $F(x)$ is defined and continuously differentiable in some neighborhood of $x = 0$, satisfies $F(0) = 0$, and is such that all eigenvalues of $(DF)_0$ have negative real part. Then $x(t) \equiv 0$ is an asymptotically stable solution of

$$x' = F(x).$$

We have already indicated in Remark 4.3(a) that we do not expect stability (in the sense of Definition 5.1) of the solution of the linear problem to imply stability for the non-linear problem. But there is a theorem corresponding to Theorem 5.2 for unstable solutions.

Theorem 5.4: *Suppose that the matrix $A(t)$ in (5.3) is an $n \times n$ constant matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$, and that at least one λ_i has positive real part. Then if the function $h(t,x)$ satisfies condition (C_η) for some sufficiently small η, $x(t) \equiv 0$ is an unstable solution of the system (5.3).*

The proof is somewhat complicated algebraically and requires reduction of A to an appropriate canonical form; we begin by defining this form and then prove that the reduction is possible.

Definition 5.2: Suppose that $\gamma \in \mathbb{R}$, $\gamma \neq 0$. An $n \times n$ matrix K is in *γ-modified real canonical form* if

$$K = \begin{bmatrix} K_1 & & & 0 \\ & K_2 & & \\ & & \ddots & \\ 0 & & & K_m \end{bmatrix}, \tag{5.7a}$$

where each block K_i has the form

$$K_i = \begin{bmatrix} \Lambda_i & \gamma I & & & 0 \\ & \Lambda_i & \gamma I & & \\ & & \Lambda_i & \ddots & \\ & & & \ddots & \gamma I \\ 0 & & & & \Lambda_i \end{bmatrix}. \tag{5.7b}$$

Here either (i) $\Lambda_i = \lambda_i$, with λ_i a real eigenvalue of K, and $I = 1$ is the 1×1 identity , or (ii) $\Lambda_i = \begin{bmatrix} \operatorname{Re}\lambda_i & \operatorname{Im}\lambda_i \\ -\operatorname{Im}\lambda_i & \operatorname{Re}\lambda_i \end{bmatrix}$, with λ_i a complex eigenvalue of K, and I is the 2×2 identity matrix.

Lemma 5.5: *Suppose that $\gamma \neq 0$. Then every $n \times n$ matrix A is similar to a matrix K in γ-modified real canonical form (5.7).*

Proof: By our construction of the real canonical form of A we know that there exists a basis of R^n composed of vectors $\{u^{ij}, v^{ij}, w^{ij}\}$ which satisfy

$$\begin{aligned} Au^{ij} &= \lambda_i u^{ij} + u^{i,j-1}, \\ Av^{ij} &= \alpha_i v^{ij} - \beta_i w^{ij} + v^{i,j-1}, \\ Aw^{ij} &= \alpha_i w^{ij} + \beta_i v^{ij} + w^{i,j-1}, \end{aligned}$$

where it is understood that the last terms are absent if $j = 1$. If we define $\tilde{u}^{ij} = \gamma^j u^{ij}$, $\tilde{v}^{ij} = \gamma^j v^{ij}$, and $\tilde{w}^{ij} = \gamma^j w^{ij}$, then multiplying these equations by γ^j yields

$$\begin{aligned} A\tilde{u}^{ij} &= \lambda_i \tilde{u}^{ij} + \gamma \tilde{u}^{i,j-1}, \\ Av^{ij} &= \alpha_i \tilde{v}^{ij} - \beta_i \tilde{w}^{ij} + \gamma \tilde{v}^{i,j-1}, \\ Aw^{ij} &= \alpha_i \tilde{w}^{ij} + \beta_i \tilde{v}^{ij} + \gamma \tilde{w}^{i,j-1}, \end{aligned}$$

and A has the form (5.7) in the new basis. ■

Our next result shows that the change of variables necessary to reduce A to canonical form does not affect the nature of the stability problem. We state it in terms of a more general, time-dependent, coordinate change, since that generality will be needed in the next section.

Lemma 5.6: *Suppose that $Q(t)$ is an $n \times n$ matrix defined on $I \supset [0, \infty)$, and that Q is continuously differentiable and has continuous inverse. Then $x(t)$ is a solution of (5.3) iff $x(t) = Q(t)y(t)$ with $y(t)$ a solution of*

$$y' = B(t)y + g(t, y), \tag{5.8}$$

where $B(t) = Q(t)^{-1}[A(t)Q(t) - Q'(t)]$ and $g(t, y) = Q^{-1}(t)h(t, Q(t)y)$. Moreover, if also $\|Q(t)\| \le M_1$ and $\|Q^{-1}(t)\| \le M_2$ for all $t \in I$, then (i) if h satisfies condition (C_η) then g satisfies condition $(\mathrm{C}_{M_1 M_2 \eta})$, and vice versa; (ii) 0 is a stable, asymptotically stable, or unstable solution of (5.3) iff 0 is a stable, asymptotically stable, or unstable solution, respectively, of (5.8).

Proof: This is a straightforward verification which we leave to the reader.

Proof of Theorem 5.4: Let $\mu > 0$ be the minimum among the positive real parts of eigenvalues of A. Set $\gamma = \mu/6$ and suppose that $A = QKQ^{-1}$ with K of the form (5.7). Then Lemma 5.6 immediately implies that it suffices to prove the theorem with A replaced by K, that is, for the special system

$$x' = Kx + h(t, x). \tag{5.9}$$

Let us label the components of x corresponding to a block K_i as $(x_{ij})_{1 \le j \le n_i}$, where $x_{ij} \in \mathbb{R}$ in case (i) of Definition 5.2 and $x_{ij} = \begin{bmatrix} x_{ij1} \\ x_{ij2} \end{bmatrix} \in \mathbb{R}^2$ in case (ii). Then for any solution $x(t)$ of (5.9),

$$x'_{ij} = \Lambda_i x_{ij} + \gamma x_{i,j+1} + h_{ij}(t, x),$$

where the term involving γ is missing if $j = n_i$. Suppose that the blocks of K are numbered so that $\operatorname{Re} \lambda_i > 0$ if $1 \le i \le m'$ and $\operatorname{Re} \lambda_i \le 0$ otherwise. If $x(t)$ is a solution of (5.9) we define

$$R^2(t) = \sum_{i=1}^{m'} \sum_{j=1}^{n_i} x_{ij}^T x_{ij} \qquad \text{and} \qquad r^2(t) = \sum_{i=m'+1}^{m} \sum_{j=1}^{n_i} x_{ij}^T x_{ij}.$$

(Note $x_{ij}^T x_{ij} = x_{ij}^2$ if $x_{ij} \in \mathbb{R}$, $x_{ij}^T x_{ij} = x_{ij1}^2 + x_{ij2}^2$ if $x_{ij} \in \mathbb{R}^2$.) The trick of the proof is to find some quantity which must increase exponentially in time; we will show that $R(t) - r(t)$ has this property.

We begin by deriving an estimate for $R'(t)$, using the formula

$$\frac{d}{dt} R^2 = 2RR' = 2 \sum_{i=1}^{m'} \sum_{j=1}^{n_i} \Big[x_{ij}^T \Lambda_i x_{ij} + \gamma x_{ij}^T x_{i,j+1} + x_{ij}^T h_{ij} \Big]. \tag{5.10}$$

We estimate the three terms in (5.10) in turn. First, it is an easy calculation to see that

$$x_{it}^T \Lambda_i x_{ij} = \operatorname{Re} \lambda_i \, x_{ij}^T x_{ij} \geq \mu x_{ij}^T x_{ij}. \tag{5.11a}$$

Second, by the Cauchy-Schwarz inequality,

$$\left| \sum_{i=1}^{m'} \sum_{j=1}^{n_i - 1} x_{ij}^T x_{i,j+1} \right| \leq \left[\sum_{i=1}^{m'} \sum_{j=1}^{n_i - 1} x_{ij}^T x_{ij} \right]^{1/2} \left[\sum_{i=1}^{m'} \sum_{j=2}^{n_i} x_{ij}^T x_{ij} \right]^{1/2} \leq R^2(t). \tag{5.11b}$$

Finally, note that for $i \leq m'$ the numbers $|x_{ij}|$, $|x_{ij1}|$, and $|x_{ij2}|$ are at most R, and that for any $y \in \mathbb{R}^n$ the Cauchy-Schwarz inequality yields $|y| \leq \sqrt{n} (\sum y_k^2)^{1/2}$. Hence condition ($C_\eta$) on h implies

$$\left| \sum_{i=1}^{m'} \sum_{j=1}^{n_i} x_{ij}^T h_{ij} \right| \leq R|h| \leq \eta R |x| \leq \eta \sqrt{n} R (R^2 + r^2)^{1/2} \leq \eta \sqrt{n} R (R + r). \tag{5.11c}$$

Inserting (5.11) into (5.10) yields $RR' \geq \mu R^2 - \gamma R^2 - \eta \sqrt{n} R (R + r)$ or

$$R' \geq (\mu - \gamma) R - \eta \sqrt{n} (R + r). \tag{5.12}$$

A very similar calculation shows that

$$r' \leq \gamma r + \eta \sqrt{n} (R + r), \tag{5.13}$$

and hence, subtracting (5.13) from (5.12), we have the estimate

$$(R - r)' \geq (\mu - \gamma - 2\eta\sqrt{n}) R - (\gamma + 2\eta\sqrt{n}) r. \tag{5.14}$$

We now verify that 0 is an unstable solution of (5.9) whenever h satisfies condition (C_η) with $\eta \leq \mu/(6\sqrt{n})$. If D_ρ is the neighborhood specified in (C_η) let $\epsilon = \rho/2$; we will show that for any $\delta > 0$ there is an x^1 with $|x^1| < \delta$ such that, if $x(t)$ is a solution of (5.9) with $x(0) = x^1$, then $|x(t)| > \epsilon$ for some $t > 0$. To see this, it suffices to choose x^1 so that $c \equiv R(0) - r(0) > 0$. Now with $\gamma = \mu/6$ and $\eta = \mu/(6\sqrt{n})$, (5.14) becomes

$$(R - r)'(t) \geq \frac{\mu}{2} (R - r)(t);$$

this equation certainly holds for those t for which $|x(s)| \leq \rho$ for $0 \leq s \leq t$. Hence, for such t,

$$R(t) - r(t) \geq (R(0) - r(0)) e^{t\mu/2} \geq c e^{t\mu/2}. \tag{5.15}$$

Now consider $t = 2 \log(\epsilon/c)/\mu$. (5.15) can be false for this t only if $|x(s)| \geq \rho > \epsilon$ for some $s \leq t$; otherwise, (5.15) is valid and implies that $|x(t)| \geq R(t) \geq (R - r)(t) \geq \epsilon$. In either case the instability of 0 is verified. ■

Remark 5.2: There is an immediate corollary of this theorem which is exactly parallel to Corollary 5.2 above; we omit a detailed statement.

5.2 Stability of periodic solutions of non-autonomous systems

In this section we again consider equation (5.1), $x' = f(t,x)$; here we study linearization near a non-constant solution. Rather than postulating directly (as we did in Section 5.1) that a linearization exists near a specific solution, we will assume that $f(t,x)$ *is continuously differentiable in the variables* x, that is, that $D_x f$ exists and is continuous in the domain D, so that we may always expand

$$f(t,x+y) = f(t,x) + D_x f_{(t,x)} y + e(t,x,y), \tag{5.16}$$

with $\lim_{y\to 0} |y|^{-1} e(t,x,y) = 0$ (see Theorem 2.14).

Suppose now that $x(t)$ is some solution of (5.1) defined on $[0,\infty)$. If we set $A(t) = D_x f_{(t,x(t))}$ and define $h(t,y) = e(t,x(t),y)$ in $D' = \{(t,y) \mid (t, y + x(t)) \in D\}$, then (5.16) becomes

$$f(t,x(t)+y) = f(t,x(t)) + A(t)y + h(t,y);$$

we know that $|h(t,y)| = o(|y|)$, although not necessarily uniformly in t. Thus if $\tilde{x}(t)$ is any other solution of (5.1), and

$$y(t) = \tilde{x}(t) - x(t),$$

then y satisfies $y'(t) = \tilde{x}'(t) - x'(t) = f(t, x(t) + y(t)) - f(t,x(t))$ or

$$y' = A(t)y + h(t,y). \tag{5.17}$$

Equation (5.17) is called the *variational equation* of (5.1), relative to the solution $x(t)$; its linearization $y' = A(t)y$ is called the *linear variational equation.* It is straightforward to verify

Lemma 5.7: *$x(t)$ is a stable solution of (5.1) in D iff 0 is a stable solution of (5.17) in D'.*

Remark 5.3: The linear variational equation $y' = A(t)y$ is essentially (2.25b), the linear differential equation we solved in Chapter II to find $J(t;T,X) \equiv D_X \hat{x}_{(t;T,X)}$. In each case, the equation may be thought of as describing the time evolution of an infinitesimal perturbation of the initial value $x(0)$. To see the connection more directly, suppose that $x(t)$ and $\tilde{x}(t)$ above correspond to $\hat{x}(t;T,X)$ and $\hat{x}(t;T,X+\epsilon Y)$, respectively. Then

$$J(t;T,X)Y = \lim_{\epsilon\to 0} \epsilon^{-1}[\hat{x}(t;T,X+\epsilon Y) - \hat{x}(t;T,X)] = \lim_{\epsilon\to 0} \epsilon^{-1} y_\epsilon(t), \tag{5.18}$$

and it follows formally from (5.17) and (5.18) that

$$J'Y = \lim_{\epsilon\to 0} \Big[A(t)\big(\epsilon^{-1} y_\epsilon(t)\big) + \epsilon^{-1} h(t, y_\epsilon(t)) \Big] = AJY,$$

since $|h(t,y_\epsilon)| = o(|y_\epsilon|) = o(\epsilon)$, and this is (2.25b).

We now consider the special case in which $f(t,x)$ is periodic in t and $x(t)$ is a periodic solution of (5.1) with rationally related period. Specifically, we assume that, for some $\tau > 0$,

$$f(t+\tau, x) = f(t,x) \qquad \text{and} \qquad x(t+\tau) = x(t),$$

and (implicitly) that $(t,x) \in D$ iff $(t+\tau,x) \in D$; τ is not necessarily the minimal period of either x or f. Then $A(t) = D_x f_{(t,x(t))}$ is τ-periodic. $h(t,y)$ in (5.17) is also periodic, and is therefore defined in some uniform neighborhood $\{(t,x) \mid |x| \le \rho\}$ of $x = 0$ and satisfies condition (C_η) for any $\eta > 0$.

Because $A(t)$ is periodic, we may apply the Floquet theory of Section 3.4 to the linear variational equation

$$y' = A(t)y. \tag{5.19}$$

Thus any fundamental matrix for (5.19) has the form $P(t)e^{tR}$, with P τ-periodic. Recall that by definition the *characteristic multipliers* of $A(t)$ are the eigenvalues of $e^{\tau R}$, or equivalently the numbers $e^{\tau\lambda}$ with λ an eigenvalue of R.

The fundamental stability result is for this type of periodic solution is

Theorem 5.8: *Let $f(t,x)$ be τ-periodic and let $x(t)$ be a τ-periodic solution of the system $x' = f(t,x)$. Then:*

(a) $x(t)$ is asymptotically stable if all characteristic multipliers of $A(t) \equiv D_x f_{(t,x(t))}$ have magnitude less than one, and

(b) $x(t)$ is unstable if at least one characteristic multiplier of $A(t)$ has magnitude greater than one.

Proof: By Lemma 5.7 it suffices to analyze (5.17). We adopt the notation from Floquet theory recalled above and introduce a new variable $z(t)$ by $y(t) = P(t)z(t)$; by Lemma 5.6, the original problem is now reduced to the study of

$$z' = P^{-1}(t)[A(t)P(t) - P'(t)]z + g(t,z), \tag{5.20}$$

where g satisfies (C_η) for any $\eta > 0$. But from $\left(P(t)e^{tR}\right)' = A(t)P(t)e^{tR}$ it follows that $P^{-1}[AP - P'] = R$, a constant matrix, so that (5.20) is of the type analyzed in section 5.1. Now the conditions on the characteristic multipliers given in (a) and (b) imply respectively that either (a) all eigenvalues of R have negative real parts, or (b) at least one eigenvalue of R has positive real part. Then Theorem 5.2, in case (a), or Theorem 5.4, in case (b), implies the result. ■

5.3 Stability of periodic solutions of autonomous systems

In this section we consider the autonomous system

$$x' = f(x), \tag{5.21}$$

where, again in order to permit linearization, we will assume that $f \in C^1(D)$ for some open, connected $D \subset \mathbb{R}^n$. The analysis of the preceding section is applicable for discussion of the stability of a critical point x^0 of this system; we must simply look at the eigenvalues of the derivative Df_{x^0}. Here we want to investigate criteria for stability of periodic solutions.

Remark 5.4: Our investigation will in fact require new definitions of stability, since some of the concepts and results of the previous sections are not useful in our current context.

(a) A periodic solution of an autonomous system can never be asymptotically stable. For if $x(t)$ is periodic (but not constant) and $x(t_0) = x^0$, then for any $\delta > 0$, $\tilde{x}(t) \equiv x(t-\delta)$ solves the IVP $\tilde{x}' = f(\tilde{x})$, $\tilde{x}(t_0) = x^1 \equiv x(t_0 - \delta)$; by choosing δ small we may make $|x^1 - x^0|$ as small as we like, but it is not true that $\lim_{t\to\infty} |\tilde{x}(t) - x(t)| = 0$.

(b) This makes it clear that Theorem 5.8(a) cannot apply in the autonomous case. We may also see directly that if $x(t)$ is a periodic, non-constant solution, then $A(t) \equiv Df_{x(t)}$ must have at least one characteristic multiplier equal to one. For differentiating $x'(t) = f(x(t))$ with respect to t yields $[x'(t)]' = Df_{x(t)}x'(t) = A(t)x'(t)$, so that $x'(t)$ is a solution of the linear variational equation (5.19). Because $P(t)e^{tR}$ is a fundamental matrix for (5.19), $x'(t) = P(t)e^{tR}c$; then since $x(t)$ and hence $x'(t)$ are periodic,

$$x'(\tau) = P(\tau)e^{\tau R}c = x'(0) = P(0)c.$$

Since $P(t)$ is also periodic this yields $e^{\tau R}c = c$, i.e., $e^{\tau R}$ has 1 as an eigenvalue.

(c) The remarks in (a) and (b) above are closely related, for the exponent found in (b) describes the behavior of infinitesimal perturbations of the initial data along the orbit itself, and its value of 1 shows that such perturbations neither shrink nor grow. The discussion in (a) describes finite perturbations of the same type, which in fact show the same behavior.

We next introduce a new type of stability which is well suited for the study of autonomous systems. For simplicity we define it only for periodic orbits.

Definition 5.3: Let $x(t)$ be a periodic solution of (5.21) and let C_p be its orbit. We say that $x(t)$ or C_p is *orbitally stable* if

(a) There exists a $b > 0$ such that, if $d(x^1, C_p) < b$, then the solution $\tilde{x}(t)$ of the IVP

$$\tilde{x}' = f(\tilde{x}), \qquad \tilde{x}(0) = x^1, \tag{5.22}$$

is defined for all $t \geq 0$.

(b) For every $\epsilon > 0$ there exists a δ, with $0 < \delta \leq b$, such that if $d(x^1, C_p) < \delta$ then the solution $\tilde{x}(t)$ of the IVP (5.22) satisfies $d(\tilde{x}(t), C_p) < \epsilon$ for all $t > 0$.

We say that $x(t)$ or C_p is *asymptotically orbitally stable* if (a) and (b) hold and if, in addition:

(c) There exists a $\bar{\delta}$, with $0 < \bar{\delta} \le b$, such that if $d(x^1, C_p) < \bar{\delta}$ then the solution $\tilde{x}(t)$ of the IVP (5.22) satisfies $\lim_{t\to\infty} d(\tilde{x}(t), C_p) = 0$.

Finally, we say that $x(t)$ or C_p is *asymptotically orbitally stable with asymptotic phase* if (a) and (b) hold and if, in addition:

(d) There exists a $\bar{\delta}$, with $0 < \bar{\delta} \le b$, such that if $d(x^1, C_p) < \bar{\delta}$ then, for some $\sigma \in \mathbb{R}$, the solution $\tilde{x}(t)$ of the IVP (5.22) satisfies $\lim_{t\to\infty} |\tilde{x}(t) - x(t+\sigma)| = 0$.

Example 5.1: To understand the force of condition (d), consider the two autonomous systems in $D = \mathbb{R}^2 \setminus \{0\}$, written in polar coordinates as

I.		II.	
	$r' = 1 - r$		$r' = (1-r)^3$
	$\theta' = r,$		$\theta' = r.$

Both systems have the periodic solution $r_p(t) = 1$, $\theta_p(t) = \theta_0 + t$, with the unit circle as periodic orbit. The general solutions, for $r(0) = r_0 \ne 1$ and $\theta(0) = \theta_0$, are

I.	II.
$r(t) = 1 - (1-r_0)e^{-t},$	$r(t) = 1 - (1-r_0)/g(t),$
$\theta(t) = \theta_0 + t - (1-r_0)(1-e^{-t}),$	$\theta(t) = \theta_0 + t + [1-g(t)]/(1-r_0),$

where $g(t) = \sqrt{1 + 2(1-r_0)^2 t}$. For each solution, $r(t) \to 1$ as $t \to \infty$, which is asymptotic orbital stability of the periodic solution. For system I there is an asymptotic phase: $\lim_{t\to\infty} |r(t) - r_p(t+\sigma)| = 0$ and $\lim_{t\to\infty} |\theta(t) - \theta_p(t+\sigma)| = 0$ for $\sigma = -(1-r_0)$. For system II, on the other hand, for any fixed σ we have $\theta(t) - \theta_p(t+\sigma) \sim \sqrt{2t}$ as $t \to \infty$, so that there is no asymptotic phase.

Now we saw in Remark 5.4 that perturbations of the initial data along the periodic orbit itself neither grow nor shrink with time. It is possible, however, that all other infinitesimal perturbations shrink, so that other solutions approach the orbit. In fact, we have

Theorem 5.9: *Suppose that $x(t)$ is a non-constant solution of (5.21) with period τ and orbit C_p, and that $n-1$ of the characteristic multipliers of $A(t) \equiv Df_{x(t)}$ have magnitude less than one. Then $x(t)$ is asymptotically orbitally stable with asymptotic period.*

Note that this theorem may be difficult to apply since, in general, we must solve the linear variational equation to obtain the characteristic multipliers of $A(t)$. This is in contrast to the theorems of Section 5.1, which depended only on a calculation of $D_x f$ at points of the solution.

We will give a proof of Theorem 5.9, based on that in Hirsch and Smale, which develops the important concept of the Poincarè map associated with a periodic orbit. We begin by defining this map.

Let C_p be the orbit of a periodic solution $x(t)$ of minimal period τ. Let $H \subset \mathbb{R}^n$ be a hyperplane $H = \{z \in \mathbb{R}^n \mid z \cdot n = C\}$ which contains the point $x^0 \equiv x(0)$ on C_p and is transverse to C_p at this point, that is, for which the normal vector n is not parallel to $f(x^0)$. Suppose that z is a point of H which is very close to x^0. Then the solution $\tilde{x}(t)$ with $\tilde{x}(0) = z$ will stay close to $x(t)$ for a long time—say, at least time 2τ if $|z - x^0|$ is small enough—and hence will intersect H again, at some time approximately equal to τ, at a point $\hat{z}$ which is also close to x^0. The mapping which takes z to $\hat{z}$ is called the Poincaré map for the hyperplane H.

We may construct the Poincarè map more carefully as follows. Let V be a (relatively) open subset of H which contains x^0 and which satisfies $f(x) \cdot n \geq a > 0$ for all $x \in V$. (V is called a *section* or *local section* of the flow Φ.) By mimicking the construction of Lemma 4.7 we may define a flow box $\phi : W \to \mathbb{R}^n$, with $W = (-\epsilon, \epsilon) \times V$ for some $\epsilon > 0$ and $\phi(t, x) = \Phi_t(x)$ for $(t, x) \in W$. In fact, the construction is considerably simplified here by the fact that f and hence ϕ are C^1; the proof that ϕ is well defined and 1–1 is the same, but the inverse function theorem now implies immediately that $\phi(W)$ is open and that ϕ^{-1} is also C^1. Since $x(\tau) \equiv \hat{x}(\tau; 0, x^0) = x^0$, continuity in initial conditions implies that there exists a (relatively) open subset $U \subset V$ with $\hat{x}(\tau; 0, z) \in \phi(W)$ for $z \in U$.

Definition 5.4: For $z \in U$, define $(s(z), g(z)) \in W$ by $(s(z), g(z)) \equiv \phi^{-1}(\hat{x}(\tau; 0, z))$. Then $T(z) \equiv \tau - s(z)$ is the *time of first return* of the point z to the hyperplane H, and $g : U \to V$ is called the *Poincaré map* for the hyperplane H.

It is clear that x^0 is a fixed point of g, i.e., that $g(x^0) = x^0$, and that $T(x^0) = \tau$. Moreover, because $\hat{x}$ and ϕ^{-1} are C^1 maps, g and T are also C^1. The motivation for the introduction of the Poincaré map is that asymptotic stability of the solution $x(t)$ should be reflected in the behavior of iterates of g: specifically, in the fact that $g^m(z)$ should approach x^0 as m increases, for any $z \in U$ sufficiently close to x^0. When this is true, we will say that x^0 is an *attracting fixed point* for g.

In fact, we will deduce Theorem 5.9 from the fact that, under the hypotheses of this theorem, x^0 is an attracting fixed point for g. To prove the latter, we will first show (Lemma ?) that x^0 is attracting if all eigenvalues of the derivative of g at x^0 are less than 1 in magnitude, then verify (Lemmas ? and ?) that the eigenvalue condition in the theorem translates immediately into an eigenvalue condition on the derivative of g. Note that this derivative Dg_z, defined as usual for $z \in H$ by

$$(Dg_z)y = \lim_{h \to 0} h^{-1}[g(z + hy) - g(z)],$$

is naturally regarded as a map from H_0 to H_0, where $H_0 = \{y \mid y \cdot n = 0\}$ is the hyperplane parallel to H through the origin.

Lemma 5.10: *Suppose that $U \subset V \subset H$ as above and that $g : U \to V$ is C^1. Let $B = Dg_{x^0}$, and suppose that all eigenvalues of B have absolute value less than 1. Then there exists a norm $\|y\|$ on H_0 and numbers $\nu < 1$ and $\delta > 0$ such that if $\|z - x^0\| < \delta$ then $z \in U$ and $\|g(z) - x^0\| \leq \nu \|z - x^0\|$. In particular, for $\|z - x^0\| < \delta$, $\|g^m(z) - x^0\| \leq \nu^m \|z - x^0\| \to 0$ as $m \to \infty$.*

In the proof we will use

Lemma 5.11: *Let C be an $n \times n$ complex matrix and let $\mu = \sup |\lambda|$, the supremum taken over all eigenvalues λ of C. Then $\mu = \lim_{n\to\infty} \|C^n\|^{1/n}$.*

The constant μ in Lemma 5.11 is called the *spectral radius* of the matrix C.

Proof: Since $\mu = |\lambda|$ for some eigenvalue λ we have $C^n x = \lambda^n x$ for x a corresponding eigenvector; hence $\|C^n\| \geq \mu^n$ and $\liminf_{n\to\infty} \|C^n\|^{1/n} \geq \mu$. Now consider the matrix function $R(z) = (C - zI)^{-1}$, which is analytic (e.g., by Cramer's rule) for z not an eigenvalue of C, in particular, for $|z| > \mu$. For $|z| > \|C\|$ we have a convergent expansion $R(z) = -\sum_{k=0}^{\infty} z^{-(n+1)} C^n$, from which we find

$$C^n = \frac{1}{2\pi i} \oint_{|z|=r} z^n (C - zI)^{-1} dz \tag{5.23}$$

whenever $r > \|C\|$. By Cauchy's formula, then, (5.23) must hold also for any $r > \mu$, from which $\|C^n\| \leq K_r r^n$ for all n (with $K_r = (r/2\pi) \sup_{|z|=r} \|C - zI)^{-1}\|$). Thus $\limsup_{n\to\infty} \|C^n\|^{1/n} \leq r$ for any $r > \mu$. ■

Proof of Lemma 5.10: Let μ be the spectral radius of B and choose γ with $\mu < \gamma < 1$. We define $\| \ \|$ by

$$\|y\| = \sum_{k=0}^{\infty} \gamma^{-k} |B^k y|;$$

it is easy to verify from Lemma 5.11 that the series converges and, from this, that $\| \ \|$ is a norm (note $\|y\| \geq |y|$). Clearly $\|By\| = \gamma \|y\|$. Finally,

$$\|g(z) - x^0\| = \|B\,(z - x^0) + o(\|z - x^0\|)\| \leq \nu \|z - x^0\|,$$

if $\gamma < \nu < 1$ and $\|z - x^0\|$ is sufficiently small. ■

The next step is to relate the eigenvalues of Dg_{x^0} to the hypotheses on the characteristic exponents of the matrix $A(t)$ given in the theorem. This is the subject of the next two lemmas.

Lemma 5.12: *The characteristic multipliers of $A(t) = Df_{x(t)}$ are the eigenvalues of $D\Phi_\tau\big|_{x^0}$, the derivative with respect to initial condition of the flow for one trip around the periodic orbit.*

Proof: Let $X(t) = D\Phi_t\big|_{x^0}$. According to (2.25), $X(t)$ satisfies the IVP

$$X'(t) = Df_{x(t)} X(t) \equiv A(t) X(t); \qquad X(0) = I.$$

$X(t)$ is thus a fundamental matrix for the linear variational equation (5.19), so that $X(t) = P(t) e^{tR}$, with P having period τ. But $X(0) = I$ implies that $P(0) = I$ and hence $X(\tau) = e^{\tau R}$. Since the characteristic multipliers of $A(t)$ are the eigenvalues of $e^{\tau R}$, the lemma is proved. ■

Now suppose that the periodic orbit $x(t)$ satisfies the hypotheses of Theorem 5.9, and let $X(t)$ be as in proof of the previous lemma. $X(\tau)$ has 1 as a simple eigenvalue, and from Remark 5.4 we see that an eigenvector is $x'(0) = f(x^0)$. Let $X(\tau) = QKQ^{-1}$ with K in real canonical form and let H_0 be the $n-1$ dimensional subspace of $\mathbb{R}^n$ spanned by the columns of Q (generalized eigenvectors) not proportional to $f(x^0)$. Finally, let H be the hyperplane through x^0 and parallel to H_0. Note that H_0 is invariant under $X(\tau)$, i.e., $X(\tau)y \in H_0$ if $y \in H_0$, and that H is transverse to $x(t)$ at $t = 0$.

Lemma 5.13: *If the Poincaré map g is defined using the hyperplane H described immediately above, then Dg_{x^0} is given by the restriction $X(\tau)\big|_{H_0}$ of $X(\tau)$ to H_0.*

Proof: For $z \in U$, $g(z) = \hat{x}(T(z); 0, z)$ satisfies $\hat{x}(T(z); 0, z) \cdot n = C$. Differentiating this equation with respect to z, applying the derivative to a vector $y \in H_0$, and evaluating at $z = x^0$ yields

$$\big[\hat{x}'(\tau; 0, x^0) \cdot n\big](DT_{x^0})y + [X(\tau)y] \cdot n = \big[f(x^0) \cdot n\big](DT_{x^0})y = 0,$$

where $X(\tau)y \cdot n = 0$, i.e., $X(\tau)y \in H_0$, because H_0 is invariant under $X(\tau)$. But $f(x^0) \cdot n \neq 0$, so

$$DT_{x^0}\big|_{H_0} = 0. \tag{5.24}$$

Then differentiating $g(z) = \hat{x}(T(z); 0, z)$ with respect to $z \in H$ yields

$$(Dg_{x^0})y = f(x^0)(DT_{x^0})y + X(\tau)y = X(\tau)y$$

for $y \in H_0$. ■

Proof of Theorem 5.9: If we define the Poincaré map g as in Lemma 5.13, then by that lemma, Lemma 5.12, and the choice of H, all eigenvalues of Dg_{x^0} are less than 1 in absolute value. Let $\|y\|$ be the norm on H_0 guaranteed by Lemma 5.10, and let $\delta > 0$ be as in that lemma, that is, such that $\|z - x^0\| < \delta$ implies $z \in U$ and $\|g(z) - x^0\| \leq \nu\|z - x^0\|$. We may suppose that δ is so small that $|T(z) - \tau| < \tau$ for $\|z - x^0\| < \delta$. For $z \in H$ with $\|z - x^0\| < \delta$ we write

$$t_m(z) = \sum_{k=0}^{m-1} T(g^k x);$$

t_m is the total time for $\hat{x}(t; 0, z)$ to make m trips around the orbit.

We first verify orbital stability, and consider initially those solutions $\tilde{x}(t) = \hat{x}(t; 0, z)$ with $z \in H$. Given $\epsilon > 0$, uniform continuity of $\hat{x}(t; 0, z)$ for $\|z - x^0\|$ small and t in a closed interval enables us to find $\delta_1 > 0$ such that, if $\|z - x^0\| < \delta_1$ and $t \in [0, 2\tau]$, then $d(\tilde{x}(t), C_p) < \epsilon$; we may assume that $\delta_1 \leq \delta$. Now suppose that $\|z - x^0\| < \delta_1$; any $t \geq 0$ may be written as $t = t_m(z) + s$ for some $m \geq 0$ and $s \in [0, 2\tau)$, so that since $\|g^m(z)\| \leq \|z\| < \delta_1$, $\tilde{x}(t) = \hat{x}(s; 0, g^m(z))$ satisfies $d(\tilde{x}, C_p) < \epsilon$.

Now consider a general solution $\tilde{x}(t) = \hat{x}(t; 0, x^1)$. Choose δ_2 so small that when $d(x^1, C_p) < \delta_2$, then, first, $d(\tilde{x}(t), C_p) < \epsilon$ for $t \in [0, 2\tau]$, and, second, $\tilde{x}(t)$ must intersect H at a point $z = \tilde{x}(t_0)$ with $t_0 \in [0, 2\tau)$ and $\|z - x^0\| < \delta_1$ (to see that this is possible one may use a flow box as in Definition 5.4). Then the argument above shows that $d(\tilde{x}, C_p) < \epsilon$ for $t \geq t_0$, and our choice of δ_2 guarantees this for $0 \leq t \leq t_0$. This completes the proof that $x(t)$ is orbitally stable.

It remains to show that $\tilde{x}(t)$ has asymptotic phase if $d(x^1, C_p)$ is sufficiently small; the argument of the previous paragraph shows that it suffices to consider solutions such that $\tilde{x}(0) = z \in H$. Suppose that $\|z - x^0\| < \delta$. Now from the definition of t_m above we may expect a total phase shift of $\tilde{x}(t)$ relative to $x(t)$, as $t \to \infty$, of

$$\sigma \equiv \lim_{m\to\infty} [m\tau - t_m(z)] = \sum_{k=0}^{\infty} [\tau - T(g^k(z))]. \tag{5.25}$$

Now

$$|T(g^k(z)) - \tau| = |T(g^k(z)) - T(x^0)| \le M\|g^k(z) - x^0\| \le M\nu^k\|z - x^0\|,$$

with M a bound on $\|DT(y)\|$ for $\|y - x^0\| < \delta$, so that σ as defined by (5.25) is finite. Then

$$\begin{aligned}\|\tilde{x}(t_m) - x(t_m + \sigma)\| &\le \|\tilde{x}(t_m) - x(m\tau)\| + \|x(m\tau) - x(t_m + \sigma)\| \\ &= \|g^m(z) - x^0\| + \|x(0) - x(t_m + \sigma - m\tau)\| \\ &\le \nu^m\|z - x^0\| + \|x(0) - x(t_m + \sigma - m\tau)\|,\end{aligned}$$

and since $\lim_{m\to\infty}(t_m + \sigma - m\tau) = 0$ and $\nu < 1$, we have $\lim_{m\to\infty} \|\tilde{x}(t_m) - x(t_m + \sigma)\| = 0$. From this it follows by continuity of $\hat{x}$ that $\lim_{t\to\infty} \|\tilde{x}(t) - x(t + \sigma)\| = 0$. ■

Example 5.2: The Van der Pol oscillator. Our discussion of the Van der Pol oscillator in Chapter IV shows that the limit cycle there is asymptotically orbitally stable; this is, in fact, a global stability property in $\mathbb{R}^2 \setminus \{0\}$, since every solution initiating in that domain tends to the limit cycle. Here we will show that this limit cycle satisfies the hypotheses of Theorem 5.9 and hence has asymptotic phase. Note that the map $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ which we defined in our previous discussion is in fact just the Poincaré map for the limit cycle, constructed using the half space $H = \{(x, y) \mid y = 0\}$ and taking $z^0 = (c^0, 0)$. Thus by Lemmas 5.12 and 5.13 we may verify that one characteristic multiplier of $A(t)$ has magnitude less than 1 by showing that $\phi'(c_0) < 1$.

Now $\phi = \psi \circ \psi$, where ψ is the map corresponding to a trip halfway around the origin, so that

$$\phi'(c_0) = \psi'(\psi(c_0))\psi'(c_0) = [\psi'(c_0)]^2.$$

On the other hand, the function $F(c) = c^2 - \psi(c)^2$ introduced in the earlier discussion satisfies

$$F'(c_0) = 2c_0 - 2\psi(c_0)\psi'(c_0) = 2c_0(1 - \psi'(c_0));$$

thus it suffices to show that $F'(c_0) > 0$ (recall that ψ is increasing so that $\psi'(c_0) > 0$). We have the decomposition $F = F_1 + F_2 + F_3$ where each of F_j is non-decreasing; hence it suffices to show that

$$F_1'(c_0) = 2\int_0^1 \frac{y^2 - y^4}{(x_{c_0}(y) + y - y^3)^2} \left.\frac{dx_c(y)}{dc}\right|_{c=c_0} dy, \tag{5.26}$$

is strictly positive. But we know that $x_c(y)$ is non-decreasing in c for $0 \le y \le 1$, and hence the integrand in (5.26) is non-negative. Moreover, $x_c\big|_{y=0} = c$ so that

$$\left.\frac{dx_c(y)}{dc}\right|_{\substack{c=c_0\\ y=0}} = 1,$$

and the integrand is strictly positive in some neighborhood of $y = 0$. This completes the verification.

It is instructive to compute the eigenvalues of

$$D_z f = \begin{bmatrix} 0 & -1 \\ 1 & 1 - 3y^2 \end{bmatrix}$$

at various points of the plane. The product of the eigenvalues is always 1. For $y^2 < 1$ the eigenvalues are complex and lie on the unit circle, with positive real part for $y^2 < 1/3$ and negative real part for $1/3 < y^2 < 1$, and for $1 < y^2$ the eigenvalues are real and negative. This means that for $y^2 < 1/3$ nearby points are pulled apart by the flow, this is basically unstable behavior. The limit cycle passes through this region; nevertheless, it also passes through the region $y^2 > 1/3$, where nearby points are pushed toward each other. The net result of these competing tendencies is shown in the characteristic multipliers of $A(t)$, which give the behavior after one complete trip around the cycle: one multiplier is 1, corresponding to no net contraction or expansion in the direction of the flow, and one is less than 1, corresponding to a contraction perpendicular to the flow.

5.4 The Second Method of Lyapunov

We begin our discussion of the second, or direct, method of Lyapunov by considering the stability of a critical point, taken by convention to be $x = 0$, of an autonomous system $x' = f(x)$, $f \in C^0(D)$. If W is a continuously differentiable scalar function defined in some neighborhood of 0 in D we define $\dot{W}(x)$ in this neighborhood by

$$\dot{W}(x) = DW_x f(x) \equiv \sum_{i=1}^{n} \frac{\partial W}{\partial x_i}(x) f_i(x),$$

so that if $x(t)$ is a solution of the system then

$$\frac{d}{dt} W(x(t)) = \dot{W}(x(t)).$$

We say that a function W defined on in a neighborhood of $x = 0$ is *positive semidefinite* if it is continuous, nonnegative, and satisfies $W(0) = 0$; it is *positive definite* if in addition $W(x) = 0$ only for $x = 0$. Negative semidefinite and negative definite functions are defined similarly. Note that if W is a quadratic form, $W(x) = x^T Bx$ for some symmetric matrix B, then this terminology corresponds with the usual notions of definite and semidefinite matrices.

The basic result of Lyapunov for this system is that:

(a) $x = 0$ is stable if there is a C^1 function $W(x)$, defined in a neighborhood of 0, such that W is positive definite and $\dot{W}$ is negative semidefinite.

(b) $x = 0$ is asymptotically stable if a W as above exists for which $\dot{W}$ is negative definite.

We will not give a formal proof of this result, which follows from the theorem for time-dependent systems which we give below, but we point out that simple geometric considerations make the result "obvious." The contours $W(x) = \lambda$ must appear as in

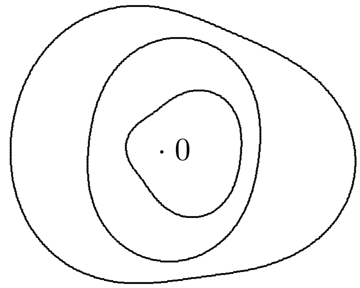


Figure 5.1

Figure 5.1 (drawn for $n = 2$); as λ decreases to 0 these contours shrink down to the origin. Because $\dot{W}$ is not positive, $W(x(t))$ is a non-increasing function of t when $x(t)$ is a solution, so that once the solution is inside some contour $W(x) = \lambda$ it can never escape; this is stability. If $\dot{W}$ is negative definite then $W(x(t))$ is strictly decreasing (assuming that $x(t)$ is not the zero solution) and thus $x(t)$ must cross all contours and approach the origin as $t \to \infty$; this is asymptotic stability.

Now we try to generalize these considerations to time-dependent systems. We will consider the equation

$$x' = f(t, x), \tag{5.27}$$

assume that f is defined and continuous in domain $D_\rho \equiv \{(t, x) \mid t \in I \text{ and } |x| < \rho\}$ for some $\rho > 0$ and $I \supset [0, \infty)$ and that $f(t, 0) \equiv 0$, and study the stability of the solution $x(t) \equiv 0$. We must introduce time-dependent Lyapunov functions $V(t, x)$ which play the role of the functions $W(x)$ in the autonomous case, and will always assume that V is defined in D_ρ (it would suffice to have V defined in $D_{\rho'}$ for some $\rho' < \rho$, but in this case we simply replace D_ρ by $D_{\rho'}$ as our fundamental domain). If $V \in C^1(D_\rho)$ we define $\dot{V} \in C^0(D_\rho)$ by

$$\dot{V}(t, x) = D_x V_{(t,x)} f(t, x) + D_t V_{(t,x)} \equiv \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(t, x) f_i(t, x) + \frac{\partial V}{\partial t}(t, x),$$

so that again, if $x(t)$ solves (5.27), then

$$\frac{d}{dt} V(t, x(t)) = \dot{V}(t, x(t)).$$

Definition 5.5: A function $V(t, x)$ defined in D_ρ is

— *positive semidefinite* if it is continuous, nonnegative, and satisfies $V(t, 0) \equiv 0$;

— *positive quasidefinite* if it is positive semidefinite and if $V(t, x) = 0$ only for $x = 0$;

— *positive definite* if it is positive semidefinite and if $V(t, x) \geq W(x)$ for some positive definite $W(x)$ defined for $\{|x| < \rho\}$.

Negative semidefinite, negative quasidefinite, and negative definite functions are defined similarly.

Remark 5.5: Note that in defining positive definite functions $V(t,x)$ we make use of our earlier definition of positive definiteness for functions of x alone. The terminology "positive quasidefinite" is not standard; it seems the most natural generalization of the earlier definition of positive definiteness for time-independent functions, but in fact is useful primarily for conceptual purposes. A positive definite function may be thought of as *uniformly* positive quasidefinite.

Theorem 5.14: *The solution $x(t) \equiv 0$ of (5.27) is*

(a) stable, if there exists a positive definite function $V \in C^1(D_\rho)$ for which $\dot{V}(t,x)$ is negative semidefinite;

(b) asymptotically stable, if a V as above exists for which $\dot{V}$ is negative definite and for which, in addition, there exists a positive definite function $W_1(x)$ defined for $\{|x| < \rho\}$ and such that $V(t,x) \le W_1(x)$ in D_ρ.

When the last condition of (b) is satisfied it is said that $V(t,x)$ has an *infinitesimal upper bound*. We will prove this theorem shortly, but first comment on the nature of the hypotheses. The obvious generalization of the autonomous result sketched above would be that stability would follow from the existence of a positive quasidefinite $V(t,x)$ with $\dot{V}(t,x)$ negative semidefinite, and asymptotic stability from the additional requirement that $\dot{V}$ be negative quasidefinite. Instead, the theorem makes additional hypotheses of uniformity: for stability, the extra hypothesis is

(U1) V is positive definite, i.e., uniformly positive quasidefinite,

and for asymptotic stability, the hypotheses are (U1) as well as

(U2) $\dot{V}$ is negative definite, i.e., uniformly negative quasidefinite,

and

(U3) V has an infinitesimal upper bound.

We will show by example that none of these uniformity hypotheses may be omitted (although it is possible that they may be replaced by alternate ones). All our examples will involve homogeneous linear equations in one unknown function ($n = 1$).

Example 5.3: (a) Consider the equation $x' = \lambda x$ for $x \in \mathbb{R}$ and $\lambda > 0$. The general solution of this equation is $x(t) = x^0 e^{\lambda t}$, so that the origin is certainly not stable. On the other hand, if $\alpha > 2\lambda$ then the function $V(t,x) \equiv x^2 e^{-\alpha t}$ is positive quasidefinite and $\dot{V}(t,x) = (2\lambda - \alpha)x^2 e^{-\alpha t}$ is negative semidefinite. Part (a) of Theorem 5.14 is not contradicted because V does not satisfy (U1).

The next two examples have the following general character: $g(t)$ is a C^1 function defined and strictly positive on some open interval I containing $[0,\infty)$, and the differential equation considered is

$$x' = \frac{g'(t)}{g(t)} x, \tag{5.28}$$

with general solution $x(t) = x^0 g(t)$.

(b) Let $g(t) = (2+t)/(1+t)$, so that 0 is a stable but not asymptotically stable solution of (5.28). Define $V(t,x) = x^2$. Then (U1) and (U3) are certainly satisfied, but

$$\dot{V}(t,x) = 2x^2 \frac{g'(t)}{g(t)} = -\frac{2x^2}{(1+t)(2+t)},$$

so that (U2) is violated.

(c) Let $G(t)$ be a strictly positive C^1 function, defined on $I \supset [0,\infty)$ and satisfying (i) $G(t) \le M$ for some $M > 0$, (ii) $G(n) \ge 1$ for all $n \in \mathbb{Z}$, $n \ge 0$, and (iii) for some $C > 0$,

$$I(t) \equiv \int_t^\infty G(s)\,ds \le CG(t).$$

Note that since $I(t)$ is finite and decreasing we must have $I(t) \to 0$ as $t \to \infty$, and there must exist a sequence $\{t_k\}$ with $t_k \to \infty$ and $G(t_k) \to 0$. G should be pictured as a rapidly decreasing function of t on which have been superimposed very thin bumps at the integers; we leave it as an exercise to verify that

$$G(t) = e^{-t} + \sum_{n=1}^{\infty} \frac{1}{1 + 10^n (t-n)^2}$$

is one possible choice. Let $g(t) = [G(t)]^{1/2}$; then again 0 is a stable but not asymptotically stable solution of (5.28). For $a \ge 0$ we define

$$V_a(t,x) = \frac{x^2}{G(t)}[a + I(t)];$$

by direct calculation we find that $\dot{V}_a(t,x) = -x^2$ so that (U2) is satisfied. If $a = 0$ then $V_0 \le Cx^2$ so that (U3) is satisfied, but $V_0(n,x) \le I(n)x^2$ so that (U1) fails. On the other hand, if $a > 0$ then $V_a(t,x) \ge ax^2/M$, so that (U1) is satisfied, but $V_a(t_n,x) \ge ax^2/G(t_n)$, so that (U3) fails.

We conclude that all uniformity hypotheses in Theorem 5.14 are necessary.

Proof of Theorem 5.14: We begin with a preliminary observation. Suppose that $W(x)$ is positive definite in $\{|x| < \rho\}$, and that $\epsilon > 0$ satisfies $\epsilon < \rho$. Then

$$\lambda(\epsilon) \equiv \inf_{\epsilon \le |x| < \rho} W(x) \qquad (5.29)$$

is strictly positive, or, more precisely, we may without loss of generality insure that $\lambda(\epsilon) > 0$ by decreasing ρ slightly if necessary. We will always assume that this has been done. Then (5.29) says that for any $\epsilon > 0$ there exists a $\lambda = \lambda(\epsilon)$ such that if $W(x) < \lambda$ (and $|x| < \rho$) then $|x| < \epsilon$.

We now prove part (a) of the theorem. Since by hypothesis $V(t,x)$ is positive definite, there exists a positive definite $W(x)$ with $W(x) \le V(t,x)$ in D_ρ. Given $\epsilon > 0$, we must find a $\delta = \delta(\epsilon) > 0$ such that, if $|x^1| < \delta$ and $x(t)$ is a solution of (5.27) with $x(0) = x^1$, defined on a maximal interval (a, b), then $b = \infty$ and $|x(t)| < \epsilon$ for all $t \geq 0$. We may suppose without loss of generality that $\epsilon < \rho$. Let $\lambda > 0$ be a number such that $|x| < \epsilon$ if $W(x) < \lambda$, and let $\delta > 0$ be chosen so that $|x^1| < \delta$ implies $V(0, x^1) < \lambda$. Since $V(0, x^1) < \lambda$ and $(d/dt)V(t, x(t)) = \dot{V}(t, x(t)) \leq 0$, we have $\lambda > V(t, x(t)) \geq W(x(t))$ and hence $|x(t)| < \epsilon$, for all $x \in [0, b)$. Now the standard argument from our extension theorem (Theorem 2.19) shows that $b = \infty$.

We now turn to part (b) of the theorem. We are given that $W(x) \leq V(t, x) \leq W_1(x)$ and that $\dot{V}(t, x) \leq -W_2(x)$ in D_ρ, for some positive definite functions W, W_1, and W_2. Take $\epsilon < \rho$; we know from (a) that there is a $\delta > 0$ such that if $|x^1| < \delta$ and $x(t)$ is a solution of (5.27) with $x(0) = x^1$ then $|x(t)| < \epsilon$ for all $t > 0$. We will verify that in this case also $\lim_{t\to\infty} x(t) = 0$. Because $\dot{V} \leq 0$, $V(t, x(t))$ is monotonic non-increasing; if $V(t, x(t))$ decreases to zero then so does $W(x(t))$, and the observation above implies that $x(t) \to 0$. Suppose then that $V(t, x(t)) \searrow \lambda > 0$. Then $W_1(x(t)) \geq \lambda$ for all t, and continuity of W_1 implies that there exists an $\epsilon' > 0$ with $|x(t)| > \epsilon'$ for all t. But now, again by the observation above, there is a λ' with $W_2(x(t)) > \lambda'$ for all t; this implies that $\dot{V}(t, x(t)) \leq -\lambda'$ for all t and hence that $V(t, x(t)) \leq V(0, x^1) - \lambda' t$, contradicting $V(t, x(t)) \geq \lambda$. ■

Remark 5.6: We can use the preceding theorem to give a new proof of Theorem 5.2, which asserted that the solution $x \equiv 0$ of

$$x' = Ax + h(t, x),$$

is asymptotically stable if all eigenvalues of A have negative real part and if $h(t, x)$ satisfies condition (C_η) for some sufficiently small η. To do so, let $-\mu < 0$ be the maximum real part of any eigenvalue, choose γ with $0 < \gamma < \mu$, let $A = QKQ^{-1}$ with K in γ-modified real canonical form, let x have coordinates x_{ij} in the basis formed by the columns of Q, and define

$$V(x) = \sum_{ij} x_{ij}^T x_{ij}.$$

V is clearly positive definite with infinitesimal upper bound (take $W = W_1 = V$ in the notation of the proof of Theorem 5.14(b)), and

$$\begin{aligned}\dot{V}(x, t) &= 2 \sum_{ij} \Big[x_{ij}^T \Lambda_i x_{ij} + \gamma x_{ij}^T x_{i,j+1} + x_{ij}^T h_{ij}(t, x) \Big] \\ &\leq -2 \big[\mu - \gamma - \eta\sqrt{n} \big] V(x),\end{aligned}$$

where we have estimated just as in the proof of Theorem 5.4. Thus if η is chosen so small that $\mu - \gamma - \eta\sqrt{n}$ is positive, then $\dot{V}$ is positive definite and Theorem 5.2 follows from Theorem 5.14.

The subject of Lyapunov's second method is a wide one, and there are many results not given here—other types of stability may be established via Lyapunov functions, the hypotheses of Theorem 5.14 may be altered, "inverse theorems" may be proved which show that a Lyapunov function must exist if stability holds, Lyapunov functions may be used to prove instability, etc. Some of these are discussed in Cronin or may be tracked down through the references there; a few are included in our problems.

5.5 Stable and Unstable Manifolds

In this section we sketch an introduction to the important topic of stable and unstable manifolds by considering the special case of a critical point of a C^1 autonomous system. As usual, we write the system as

$$x' = f(x), \tag{5.30}$$

and for simplicity take the critical point to be $x = 0$. We will suppose throughout that $x = 0$ is a *hyperbolic* critical point (see Remark 4.3), that is, that $A \equiv Df_0$ has no eigenvalues with real part zero. We let k denote the total multiplicity of all eigenvalues with negative real part, so that by hyperbolicity $n-k$ is the total multiplicity of eigenvalues with positive real part.

We begin by discussing the stable and unstable manifolds for the linearized system

$$x' = Ax. \tag{5.31}$$

We decompose $\mathbb{R}^n$ into the (direct) sum of two subspaces, writing $\mathbb{R}^n = L_s \oplus L_u$, where L_s is the k dimensional subspace spanned by all eigenvectors and generalized eigenvectors of A corresponding to eigenvalues of negative real part, and L_u, of dimension $n-k$, is defined similarly for the remaining eigenvalues. L_s and L_u are invariant under A and hence under e^{tA}, that is, since $\hat{x}(t; t_0, x^0) = e^{(t-t_0)A}x^0$, they are invariant sets for (5.31) in the sense of Chapter IV. If $x^0 \in L_s$ then $\lim_{t\to\infty} \hat{x}(t; 0, x^0) = 0$; on the other hand, if $x^0 \notin L_s$, then the expansion of x^0 in generalized eigenfunctions of A will contain some generalized eigenfunctions for eigenvalues with positive real part, so that $\lim_{t\to\infty} |x(t)| = \infty$. Thus

$$L_s = \{x^0 \in \mathbb{R}^n \mid \lim_{t\to\infty} \hat{x}(t; 0, x^0) = 0\}. \tag{5.32a}$$

For this reason L_s is called the *stable subspace* or *stable manifold* for the origin in the system (5.31). Similarly, L_u may be characterized by

$$L_u = \{x^0 \in \mathbb{R}^n \mid \lim_{t\to-\infty} \hat{x}(t; 0, x^0) = 0\}, \tag{5.32b}$$

and is called the *unstable manifold*.

Remark 5.7: (a) Note that L_u is *not* the set of initial conditions x^0 for which $\hat{x}(t; 0, x^0)$ is unbounded as $t \to \infty$; the latter set is just the complement of L_s and is not a linear subspace. Initial conditions which do not lie in either L_s or L_u lead to trajectories which become unbounded both as t approaches ∞ and as t approaches $-\infty$.

(b) Figure 5.2(a) shows a typical configuration of L_s and L_u in the case $n = 2$, $k = 1$; this is just the saddle point we studied in Section 4.2. Figure 5.2(b) illustrates the case $n = 3$, $k = 2$. In this latter case, the configuration within L_s might equally well be a stable node rather than the stable spiral shown.

We now ask to what extent this picture persists in the full system (5.30). The answer is that near the critical point 0 it survives with one minor change—the linear subspaces L_s and L_u are replaced by more general *manifolds* of the same dimension. To describe these

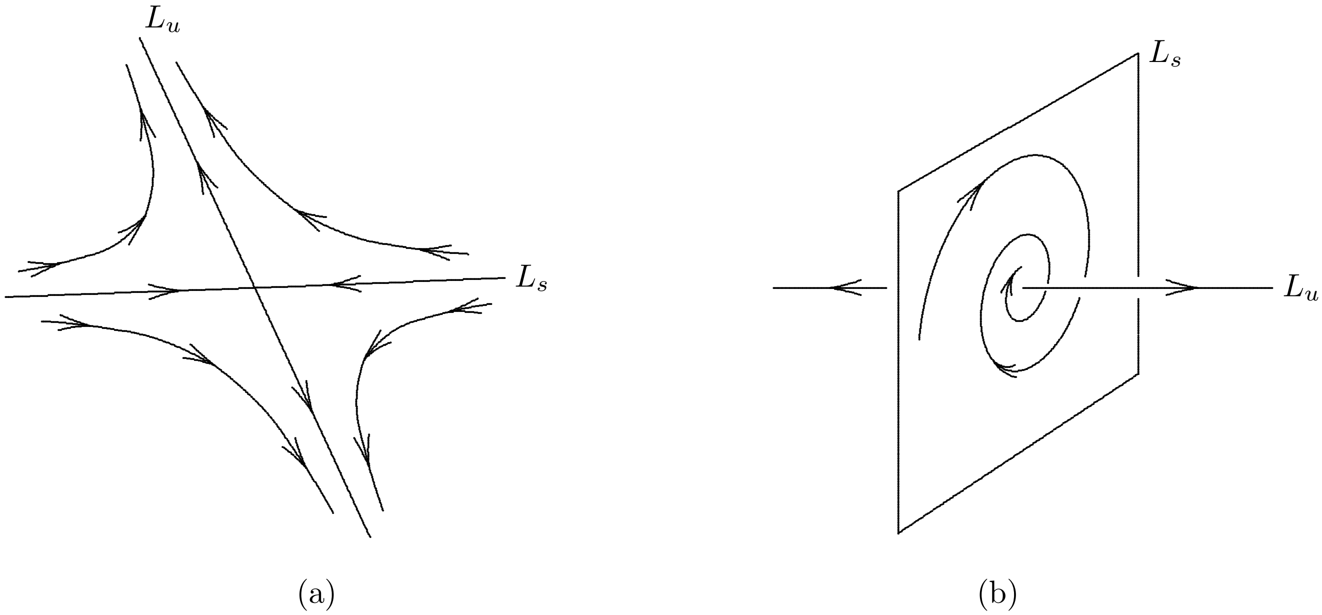


Figure 5.2: Stable and unstable subspaces.

manifolds it is convenient to use a special norm $\|\cdot\|$ in $\mathbb{R}^n$, which we will define shortly. Then for $\epsilon > 0$ we define local stable and unstable sets, W_s^ϵ and W_u^ϵ respectively, for the critical point 0, by

$$W_s^\epsilon = \{x^0 \in \mathbb{R}^n \mid \|\hat{x}(t;0,x^0)\| \le \epsilon \text{ for } t \ge 0, \text{ and } \lim_{t\to\infty} \hat{x}(t;0,x^0) = 0\}, \quad (5.33a)$$
$$W_u^\epsilon = \{x^0 \in \mathbb{R}^n \mid \|\hat{x}(t;0,x^0)\| \le \epsilon \text{ for } t \le 0, \text{ and } \lim_{t\to-\infty} \hat{x}(t;0,x^0) = 0\}. \quad (5.33b)$$

Our main theorem describes these sets very explicitly. To express it we introduce the following notation: since we know that each $x \in \mathbb{R}^n$ may be written uniquely as $x = y + z$ with $y \in L_s$ and $z \in L_u$, we treat y and z as coordinates and write $x = (y, z)$. We will write $B_\epsilon \equiv \{x \in \mathbb{R}^n \mid \|x\| \le \epsilon\}$.

Theorem 5.15: *For sufficiently small ϵ there exists a C^1 mapping $\phi : B_\epsilon \cap L_s \to L_u$, with $\phi(0) = 0$ and $D\phi_0 = 0$, such that*

$$W_s^\epsilon = \{(y, \phi(y)) \mid y \in B_\epsilon \cap L_s\}. \quad (5.34a)$$

Similarly, there exists a $\psi : B_\epsilon \cap L_u \to L_s$ with $\psi(0) = 0$ and $D\psi_0 = 0$, such that

$$W_u^\epsilon = \{(\psi(z), z) \mid z \in B_\epsilon \cap L_u\}. \quad (5.34b)$$

Remark 5.8: (a) Theorem 5.15 essentially presents the local stable manifold as the *graph* of the C^1 function ϕ. This graph is a surface or manifold of the same dimension, k, as L_s. Because $\phi(0)$ and $D\phi_0$ vanish, the manifold is tangent to L_s at the origin. The situation is illustrated in Figure 5.3. Similar observations apply to the unstable manifold. Thus the

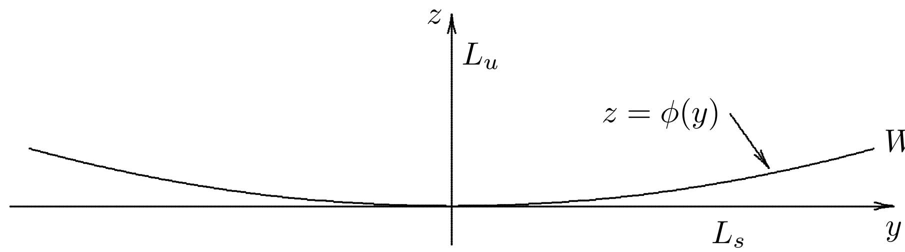
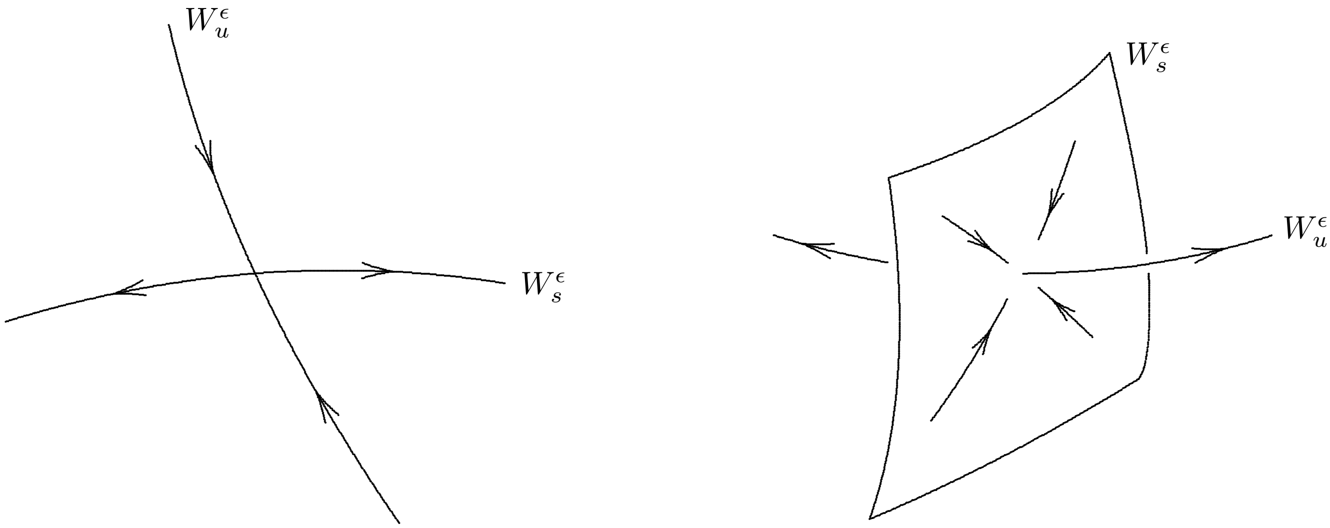


Figure 5.3. The local stable manifold as a graph.



(a) (b)

Figure 5.4: Local stable and unstable manifolds.

typical pictures of Figure 5.2 are modified for a general autonomous system as shown in Figure 5.4. We may also think of the set W_s^ϵ as the zero set of the C^1 function $F : B_\epsilon \to L_u$, defined by $F(y, z) = z - \phi(y)$.

(b) We may define global stable and unstable sets by

$$W_s = \{x^0 \in \mathbb{R}^n \mid \lim_{t\to\infty} \hat{x}(t; 0, x^0) = 0\},$$
$$W_u = \{x^0 \in \mathbb{R}^n \mid \lim_{t\to-\infty} \hat{x}(t; 0, x^0) = 0\}.$$

However, these can be much more complicated than the local sets. For example, in the case shown in Figure 5.5, W_s and W_u are identical! In higher dimensions the global picture may be extremely difficult to unravel.

Before proving the theorem we must define the norm $\|\cdot\|$; we begin by introducing some additional notation. We let P_s and P_u be the projections of $\mathbb{R}^n$ onto L_s and L_u, respectively; in the notation above, $P_s x = (y, 0)$ and $P_u x = (0, z)$ for $x = (y, z)$. P_u and P_s are linear maps of $\mathbb{R}^n$ to itself which commute with both A and e^{tA} and satisfy $P_s^2 = P_s$, $P_u^2 = P_u$ and $P_s + P_u = I$. We write $e^{tA} = Y_s(t) + Y_u(t)$, where

$$Y_s(t) = P_s e^{tA} P_s = e^{tA} P_s = P_s e^{tA} \qquad \text{and} \qquad Y_u(t) = P_u e^{tA} P_u = e^{tA} P_u = P_u e^{tA}.$$

Note that $Y_s'(t) = AY_s(t)$ and $Y_s(t)Y_s(t') = Y_s(t+t')$, and that the same equations hold for Y_u.

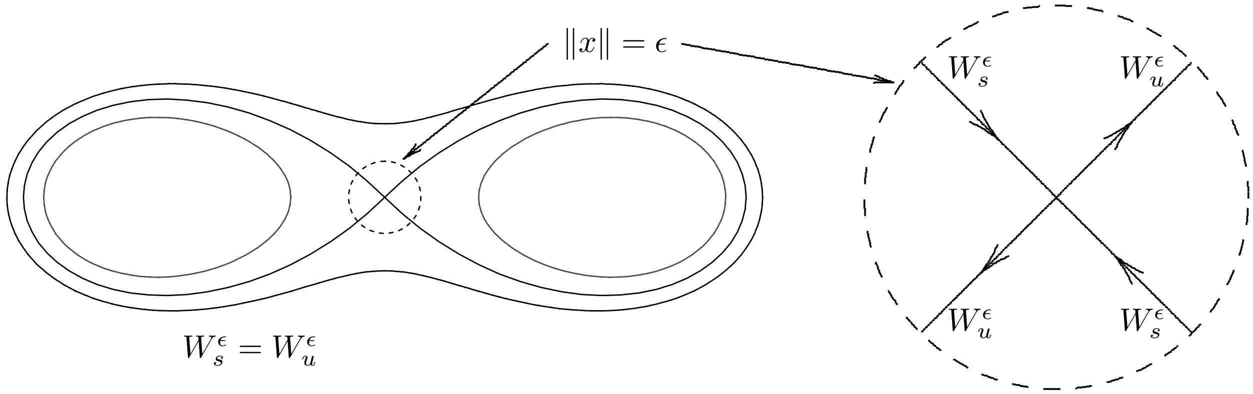


Figure 5.5. The disk $\|x\| < \epsilon$ is shown in expanded view at the right.

Hyperbolicity of the fixed point implies that, for some C and $\gamma > 0$, $|Y_s(t)x| \leq Ce^{-t\gamma}|x|$, if $t \geq 0$, and $|Y_u(t)x| \leq Ce^{t\gamma}|x|$, if $t \leq 0$, (see the proof of Theorem 5.2). We choose ν with $0 < \nu < \gamma$ and define norms $\|\cdot\|_s$ and $\|\cdot\|_u$ on L_s and L_u, respectively, by

$$\|y\|_s = \int_0^\infty e^{\nu\tau}|Y_s(\tau)y|\,d\tau, \qquad \|z\|_u = \int_0^\infty e^{\nu\tau}|Y_u(-\tau)z|\,d\tau;$$

it is elementary to verify that these are norms and that

$$\begin{aligned} \|Y_s(t)x\| &\leq e^{-t\nu}|x|, \qquad \text{if } t \geq 0; \\ \|Y_u(t)x\| &\leq e^{t\nu}|x|, \qquad \text{if } t \leq 0. \end{aligned} \tag{5.35}$$

Finally, we define $\|x\| = \max\{\|P_s x\|_s, \|P_u x\|_u\}$. We will now let $\|T\|$ denote the operator norm on $n \times n$ matrices which is defined using the norm $\|\cdot\|$ on $\mathbb{R}^n$.

Proof of Theorem 5.15: We give only an outline of the proof of the existence of the map ϕ defining the local stable manifold.

(i) Let $f(x) = Ax + g(x)$; then g is C^1 and satisfies $g(0) = 0$, $D_g(0) = 0$. By the variation of parameters formula (3.7) and (3.8), a solution $x(t)$ of (5.30) will satisfy

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}g(x(\tau))\,d\tau. \tag{5.36}$$

Suppose now that $x(t)$ is defined and satisfies $\|x(t)\| < \epsilon$ for all $t \geq 0$ and some appropriately small ϵ; in particular, this will be true if $x(0) \in W_s^\epsilon$. Then if we write $x(0) = (y, z)$ and $e^{tA} = Y_s(t) + Y_u(t)$, (5.36) becomes

$$\begin{aligned} x(t) = Y_s(t)y &+ \int_0^t Y_s(t-\tau)g(x(\tau))\,d\tau - \int_t^\infty Y_u(t-\tau)g(x(\tau))\,d\tau \\ &+ Y_u(t)\left[z + \int_0^\infty Y_u(-\tau)g(x(\tau))\,d\tau\right]. \end{aligned} \tag{5.37}$$

From (5.35) and the boundedness of $x(t)$ it is clear that the first three terms in (5.37) are bounded in t. On the other hand, because Y_u contains only growing exponentials, the last term cannot be bounded unless the quantity in brackets (which is independent of t) vanishes. We conclude: if $(y, z) \in W_s^\epsilon$ for sufficiently small ϵ, then the solution of (5.30) with initial value (y, z), a solution which we now write as $x(t; y)$, must satisfy the integral equation

$$x(t; y) = Y_s(t)y + \int_0^t Y_s(t-\tau)g(x(t; y))\,d\tau - \int_t^\infty Y_u(t-\tau)g(x(t; y))\,d\tau. \tag{5.38}$$

Moreover, z must satisfy $z = \phi(y)$, where

$$\phi(y) = -\int_0^\infty Y_u(-\tau)g(x(\tau; y))\,d\tau. \tag{5.39}$$

(ii) Next we show, by straightforward application of the method of successive approximations, that for and $\|y\| < \epsilon$ with $\epsilon > 0$ sufficiently small, (5.38) has a unique solution $x(t; y)$ satisfying $\|x(t; y)\| < \epsilon$ for all $t \geq 0$. Moreover, $x(t; y)$ solves the original system (5.30), is continuous in y, is exponentially decreasing according to

$$\|x(t; y)\| \leq \|y\|e^{-t\nu/2}, \tag{5.40}$$

and satisfies the initial condition $x(0; y) = (y, \phi(y))$, where $\phi(y)$ is given by (5.39) (note that this last conclusion follows directly from (5.38)). This verifies (5.34a). Moreover, $x(t; 0) \equiv 0$ by inspection, so that $\phi(0) = 0$.

(iii) We next prove that ϕ is C^1; since $\phi(y) = x(0; y)$, it certainly suffices to show that $x(t; y)$ is C^1 in y. To verify this, we mimic the proof of Theorem 2.15 (differentiability in initial conditions). That is, we first derive, by formal differentiation of (5.38), the integral equation which should be satisfied by $D_y x_{(t;y)}$, then show, by successive approximations, that this integral equation has a continuous, exponentially decreasing solution, say $K(t; y)$. The last step is to show that

$$\lim_{v\to 0} \|v\|^{-1}\Big[x(t; y+v) - x(t, v) - K(t; y)v\Big] = 0 \tag{5.41}$$

(Gronwall's inequality, which we used at this stage in the earlier proof, is not available here, but a somewhat similar trick works). Equation (5.41) implies that $D_y x = K$.

(iv) Finally, it follows from (5.39), (5.40), and $\|g(x)\| = o(\|x\|)$ that $D\phi_0 = 0$. ■

Appendix to Chapter V

Inspection of the references for this course reveals various definitions of stability and asymptotic stability, differing in the permitted choices of time at which to specify the initial closeness of solutions. Moreover, some authors define a concept of *uniform stability* or define various sorts of uniform asymptotic stability; in these definitions, some uniformity in the initial time or initial condition is imposed on the degree of initial closeness required or on the rate of convergence for asymptotic stability. In this appendix we summarize some of the possibilities. The discussion is based in part on: José L. Massera, "Contributions to Stability Theory,", *Ann. Math.* **64**, 1956, 182–206.

Throughout this appendix we take $x(t)$ to be a solution of (5.1) defined on some open interval which contains $[\tau, \infty)$. In particular, this implies that the domain D contains an open neighborhood of the trajectory $\{(t, x(t)) \mid t \geq \tau\}$. We will on occasion refer to various possible additional hypotheses:

(U) $f(t,x)$ is such that solutions of (5.1) are unique.

(N) D contains a uniform neighborhood $\{(t,x) \mid t \geq \tau, |x - x(t)| < \rho\}$ of $\{(t, x(t))\}$.

(P) $f(t,x)$ is periodic in t.

(A) The system is autonomous: $f(t,x)$ is independent of t.

(C) The solution $x(t)$ is constant.

(AC) Both (A) and (C) hold, i.e., the solution is a critical point of an autonomous system.

The various stability definitions we will discuss are given in the boxed display on the next page. (Notation: for $(t_0, x^0) \in D$ we let $\bar{x}(t; t_0, x^0)$ denote any solution of (5.1) satisfying $\bar{x}(t_0; t_0, x^0) = x^0$ and defined on a maximal interval; under assumption (U), $\hat{x}(t; t_0, x^0)$ is defined and $\bar{x}(t; t_0, x^0) = \hat{x}(t; t_0, x^0)$.) Roughly speaking, (i) and (iii) are the definitions of stability and asymptotic stability that we have used. (ii) adds to the stability definition the condition that δ may be chosen independently of the time t_0 at which initial conditions are imposed. (iv)–(vi) add uniformity conditions to the definition of asymptotic stability: (iv) requires that $\bar{\delta}$ may be chosen independently of t_0, (v) that the rate of decay to $x(t)$ be independent of the initial condition x^1, and (vi) that (iv) and (v) hold and that the decay rate also be independent of t_0. (vii) requires that the convergence to $x(t)$ be exponential and that all choices be uniform.

Remark 5.9: (a) It is easy to verify the implications

(i.c)	$\Rightarrow$	(i.b)	$\Rightarrow$	(i.a)
(ii.c)	$\Rightarrow$	(ii.b)	$\Rightarrow$	(ii.a)
(iii.c)	$\Rightarrow$	(iii.b)	$\Rightarrow$	(iii.a)
(iv.b)	$\Rightarrow$	(iv.a)		

Under the uniqueness assumption (U), continuous dependence on parameters implies that all these implications become equivalences. We will generally make this assumption in what follows, and therefore refer simply to (i), (ii), (iii), and (iv). The distinctions have been introduced here primarily because different authors give the definitions in different forms.

(b) All of (ii), (iv), (vi), or (vii) imply assumption (N).

STABILITY DEFINITIONS

(i) *Stability*: (a) For any $\epsilon > 0$ there exists a $t_0 \geq \tau$ and a $\delta = \delta(\epsilon, t_0) > 0$ such that, if $|x^1 - x(t_0)| < \delta$, then $\bar{x}(t; t_0, x^1)$ is defined for $t \geq t_0$ and satisfies $|\bar{x}(t; t_0, x^1) - x(t)| < \epsilon$ there.

(b) For any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that, if $|x^1 - x(\tau)| < \delta$, then $\bar{x}(t; \tau, x^1)$ is defined for $t \geq \tau$ and satisfies $|\bar{x}(t; \tau, x^1) - x(t)| < \epsilon$ there.

(c) For any $\epsilon > 0$ and for all a $t_0 \geq \tau$ there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that, if $|x^1 - x(t_0)| < \delta$, then $\bar{x}(t; t_0, x^1)$ is defined for $t \geq t_0$ and satisfies $|\bar{x}(t; t_0, x^1) - x(t)| < \epsilon$ there.

(ii) *Uniform stability*: (a) For any $\epsilon > 0$ there exists a $t_\epsilon \geq \tau$ and a $\delta = \delta(\epsilon) > 0$ such that, if $|x^1 - x(t_0)| < \delta$ and $t_0 \geq t_\epsilon$, then $\bar{x}(t; t_0, x^1)$ is defined for $t \geq t_0$ and satisfies $|\bar{x}(t; t_0, x^1) - x(t)| < \epsilon$ there.

(b) There exists a $T > \tau$ such that for any $\epsilon > 0$ and any $t_0 \geq T$ there exists a $\delta = \delta(\epsilon) > 0$ such that, if $|x^1 - x(t_0)| < \delta$, then $\bar{x}(t; t_0, x^1)$ is defined for $t \geq t_0$ and satisfies $|\bar{x}(t; t_0, x^1) - x(t)| < \epsilon$ there.

(c) For any $\epsilon > 0$ and for all a $t_0 \geq \tau$ there exists a $\delta = \delta(\epsilon) > 0$ such that, if $|x^1 - x(t_0)| < \delta$, then $\bar{x}(t; t_0, x^1)$ is defined for $t \geq t_0$ and satisfies $|\bar{x}(t; t_0, x^1) - x(t)| < \epsilon$ there.

(iii) *Asymptotic Stability*: (a) (i.a) is satisfied and (with t_0 from (i.a)) there exists a $\bar{\delta}(t_0) > 0$ such that, if $|x^1 - x(t_0)| < \bar{\delta}$, then $\lim_{t\to\infty} |\bar{x}(t; t_0, x^1) - x(t)| = 0$.

(b) (i.b) is satisfied and there exists a $\bar{\delta} > 0$ such that, if $|x^1 - x(\tau)| < \bar{\delta}$, then $\lim_{t\to\infty} |\bar{x}(t; \tau, x^1) - x(t)| = 0$.

(c) (i.c) is satisfied and, for any $t_0 \geq 0$, there exists a $\bar{\delta}(t_0) > 0$ such that, if $|x^1 - x(t_0)| < \bar{\delta}$, then $\lim_{t\to\infty} |\bar{x}(t; t_0, x^1) - x(t)| = 0$.

(iv) (a) (ii.a) is satisfied and there exists a $\bar{\delta}$ such that if $t_0 \geq t_\epsilon$ and $|x^1 - x(t_0)| < \bar{\delta}$, then $\lim_{t\to\infty} |\bar{x}(t; t_0, x^1) - x(t)| = 0$.

(b) (ii.c) is satisfied and there exists a $\bar{\delta}$ such that if $t_0 \geq 0$ and $|x^1 - x(t_0)| < \bar{\delta}$, then $\lim_{t\to\infty} |\bar{x}(t; t_0, x^1) - x(t)| = 0$.

(v) *Equiasymptotic stability* There is a $\bar{\delta} > 0$ and, for each $\epsilon > 0$, a T_ϵ such that if $|x^1 - x(\tau)| < \bar{\delta}$ and $t \geq T_\epsilon$, then $|\bar{x}(t; \tau, x^1) - x(t)| < \epsilon$.

(vi) *Uniform asymptotic stability* (ii.c) holds, and there exists a $\bar{\delta} > 0$ and, for each $\epsilon > 0$, an S_ϵ such that if $t_0 \geq \tau$, $|x^1 - x(t_0)| < \bar{\delta}$, and $t \geq t_0 + S_\epsilon$, then $|\bar{x}(t; \tau, x^1) - x(t)| < \epsilon$.

(vii) *Exponential asymptotic stability* There exists a $\mu > 0$ and, for each $\epsilon > 0$, a $\delta = \delta(\epsilon) > 0$, such that if $t_0 \geq \tau$ and $|x^1 - x(t_0)| < \delta$, then $|\bar{x}(t; \tau, x^1) - x(t)| < \epsilon \exp[-\mu(t - t_0)]$.

(c) The conditions (i) through (vii) are related by

$$\begin{array}{ccccccc} & & & \Rightarrow & \text{(v)} & \Rightarrow & \\ \text{(vii)} & \Rightarrow & \text{(vi)} & & & & \text{(iii)} \\ & & & \Rightarrow & \text{(iv)} & \Rightarrow & \\ & & \Downarrow & & & & \Downarrow \\ & & \text{(ii)} & & \Longrightarrow & & \text{(i)} \end{array}$$

These implications are quite straightforward to verify. It is instructive to construct examples showing that, in general, the reverse implications fail.

(d) Massera points out that there are further implications among these conditions when f satisfies additional assumptions. For example, under assumption (P) or assumption (A), (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (vi).

A summary of the definitions given by authors in our references is given on the following page.

Author	Terminology	Page	Definition	Assumptions
Arnold	Stability	155	(i.b)	(AC)(U)
	Asymptotic stability	156	(iii.b)	(AC)(U)
Birkhoff and Rota	Stability	121	(i.b)	(AC)(U)
	Strict stability	121	(iii.b)	(AC)(U)
Coddington and Levinson	Stability	314	(i.b)	
	Asymptotic stability	314	(iii.b)	
Cronin	Stability	151	(i.a)	
	Uniform stability	179	(ii.b)	(A)
	Asymptotic stability	151	(iii.a)	
Hale	Stability	26	(i.c)	(N)
	Uniform stability	26	(ii.c)	(N)
	Asymptotic stability	26	(iii.b)	(N)
	Uniform asymptotic stability	26	(vi)	(N)
Hartman	Stability	40	(ii.a)	(N)(C)
	Uniform stability	40	(ii.c)	(N)(C)
	Asymptotic stability	40	(iv.a)	(N)(C)
	Uniform asymptotic stability	40	(iv.b)	(N)(C)
Hirsch and Smale	Stability	185	(i.b)	(AC)(U)
	Asymptotic stability	186	(iii.b)	(AC)(U)
Lefschetz	Stability	78	(i.c)	(U)(N)(C)
	Uniform stability	78	(ii.c)	(U)(N)(C)
	Asymptotic stability	78	(iii.b)	(U)(N)(C)
	Uniform asymptotic stability	78	(vi)	(U)(N)(C)
	Stability	83	(ii.c)	(U)
	Asymptotic stability	83	(iv.b)	(U)
	Equiasymptotic stability	85	(v)	(U)(N)(C)
	Exponential asymptotic stability	85	(vi)	(U)(N)(C)
Petrovski	Stability	151	(i.b)	(N)
	Asymptotic stability	151	(iii.b)	(N)

Note: (a) Definitions related to orbital stability are not included; there may be other omissions.

(b) An assumption that $f(t, x)$ is differentiable is indicated here by the weaker assumption (U). Recall that (AC) $\Rightarrow$ (N); otherwise, (N) is indicated explicitly when mentioned by the author.