642:515 ODE Problem Set 1

Note: if you took your undergraduate ODE course(s) very long ago, **STOP.** *It is strongly suggested that, before attempting these problems, you first work out Problem Set 0, perhaps after reviewing standard techniques such as separation of variables, using any undergraduate textbook as a reference.*

1. For each of the following initial value problems, find the largest interval $I_{(t_0,x^0)}$ of existence. Justify your answers. (Note: you may use without proof any result proved in class, but you must cite the result carefully. In many cases, the answer to the problem can be given without first computing a solution.)

1. $x' = x^3\,, \quad x(0) = 0$
2. $x' = x^3\,, \quad x(0) = 1$
3. $x' = -x^3\,, \quad x(0) = 1$
4. $x' = x^3 \sin x\,, \quad x(0) = 1$
5. $x' = y^3\,,\ y' = \sin y\,, \quad x(0) = 1\,,\ y(0) = 1$

2. Assume that Picard iteration is used in order to solve $x' = x\,,\ x(0) = 1$ on the interval $[0, 1]$. Compute the iterates $x_k(t)$.

3. The Lipschitz condition in the Picard theorem is necessary: Let $n = 1$ and define

$$f(t,x) = \begin{cases} 0, & \text{if } t \le 0, \\ 2t, & \text{if } t > 0 \text{ and } x \le 0, \\ 2t - 4x/t, & \text{if } t > 0 \text{ and } 0 < x < t^2, \\ -2t, & \text{if } t > 0 \text{ and } x \ge t^2. \end{cases}$$

Show (briefly) that f is continuous but not locally Lipschitzian. Show that the method of successive approximations (i.e., the method of Picard iteration) applied to the problem $x' = f(t,x)$, $x(0) = 0$ produces a sequence of approximations $\{x_m\}$ which is not convergent, and for which no subsequence converges to a solution of the problem.

4. In proving existence of solutions of the problem $x' = f(t,x)$, $x(t_0) = x^0$, with no Lipschitz condition on f, we applied Euler's method to produce a sequence of approximate solutions $\{x_m(t)\}$.

a. Give an example of a function $f(t,x)$ and an initial point (t_0, x^0) such that $\{x_m\}$ is a convergent sequence.

b. (b) Give an example of a function $f(t,x)$ and an initial point (t_0, x^0) such that $\{x_m\}$ is not a convergent sequence.

5. Recall that a map $f : D \to \mathbb{R}$, where $D \subset \mathbb{R} \times \mathbb{R}^n$ is open, is said to satisfy a *Lipschitz condition in* x on a set $K \subset D$ if there exists a constant k_K such that

$$|f(t,x^1) - f(t,x^2)| \le k_K |x^1 - x^2|$$

whenever $(t,x^1), (t,x^2) \in K$, and that f satisfies a *local Lipschitz condition in* x on D if it satisfies a Lipschitz condition in x on each compact subset K of D. *Prove:* If f satisfies a local Lipschitz condition in x and $f(t,x)$ is continuous in t for each fixed x, then f is jointly continous in x and t.

6. Take D and f as above. Suppose that f is bounded and that for each $(t_0, x^0) \in D$, f satisfies a Lipschitz condition in x on some rectangle $R \subset D$ of the form

$$R = \{(t,x) \mid |t - t_0| \le a, |x - x^0| \le b\},$$

where $a, b > 0$. Prove that f satisfies a local Lipschitz condition in x on D.

7. Give a sketch of a proof of the local existence theorem, for the Lipschitz case, using the Contraction Mapping Theorem. (The complete metric space in which you will work will be a ball of small radius, in the space of continuous functions defined on a small time interval around t_0, around the constant trajectory x^0.) Compare the estimate that you get for the domain of solution when using this technique with the estimate obtained in the proof based on Picard iteration.

8. (Cascaded systems.) Suppose that $f(t, x)$ is continuous and locally Lipschitzian in x on the domain $D \subset \mathbb{R} \times \mathbb{R}^n$ and that $g(t, x, y)$ is continuous and locally Lipschitzian in y on $D \times U$, where $U \subset \mathbb{R}^m$ is open and connected. Prove that solutions of the IVP

$$x' = f(t, x), \qquad y' = g(t, x, y), \qquad (x(t_0), y(t_0)) = (x^0, y^0),$$

exist and are unique for any $(t_0, x^0, y^0) \in D \times U$. Show by example that our uniqueness theorem as stated and proved in class may not apply to this IVP.

9. Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and satisfy $|f(t, x) - f(t, y)| \le h(t)|x - y|$, where h is continuous. Prove that every IVP $x' = f(t, x)$, $x(t_0) = x^0$, has a unique solution defined on all of $\mathbb{R}$. (Hint: just edit the proof of the extra result proved in class for the globally Lipschitz case.)

10. Suppose that $f(t, x)$ is defined and continuous in some neighborhood of the origin in $\mathbb{R} \times \mathbb{R}$. We are interested in the IVP $x' = f(t, x)$, $x(0) = 0$. Prove that on some interval I containing the origin there exist *maximum* and *minimum* solutions $x_{\max}(t)$ and $x_{\min}(t)$ such that every solution $x(t)$ satisfies $x_{\min}(t) \le x(t) \le x_{\max}(t)$. (Hint: define $x_{\max}(t) = \sup x(t)$, the supremum taken over all solutions. Then prove that this defines a solution.)

11. Show how to derive the existence theorem for the continuous case (exactly as stated, with the same domain of existence) from the one for the Lispchitz case, by approximating the right hand side of the equation by a Lipschitz function and then using Ascoli-Arzela.