642:515 ODE Problem Set 2

The objective of this assignment is to clarify what is *not* implied by the continuous dependence theorem.

Consider the following system of equations ("Lorenz system"):

$$\begin{aligned} \dot{x}_1 &= 10(x_2 - x_1) \\ \dot{x}_2 &= 28x_1 - x_2 - x_1x_3 \\ \dot{x}_3 &= -\frac{8}{3}x_3 + x_1x_2\,. \end{aligned}$$

Since the vector field is polynomial, it is of course locally Lipschitz.

1. Suppose that we know that certain trajectories stay in the cube given by the inequalities $-20 \leq x_i \leq 20$, $i = 1, 2, 3$. *Find explicitly* a constant c such that $|\varphi(0.01, 0, x) - \varphi(0.01, 0, x')| \leq c|x - x'|$ for all vectors x and x' in this cube. (Hint: find the Jacobian of f, and estimate its norm; then use Gronwall.)

Next, consider the solutions $x(t)$ and $x'(t)$ corresponding to the two initial conditions $x = (0, 1, 0)$ and $x' = (0, 1.001, 0)$ respectively. Calculate, using a numerical differential equation solver, the difference $x_1(0.01) - x'_1(0.01)$ between the first coordinates of the two solutions, at time $t = 0.01$. (Use any package you want: Maple, Matlab, Mathematica, ...) Check that the initial condition error doesn't "get amplified" by more than the c which you obtained.

2. Now find a constant c that works for $\varphi(10, 0, x)$.

Consider again the solutions $x(t)$ and $x'(t)$ corresponding to the two initial conditions $x = (0, 1, 0)$ and $x' = (0, 1.001, 0)$ respectively. Calculate, using a numerical equation solver, the difference $x_1(10) - x'_1(10)$ corresponding to the two solutions at time 10. You should get an error of less than 10^{-3}. Conclude that your estimate was overly conservative. Show, on the same figure, the plots of $x(t)$ and $x'(t)$, for $0 \leq t \leq 10$.

3. Now repeat for the interval $0 \leq t \leq 100$ (with the same initial conditions). Explain what you see and why it does not (or does it?) contradict the conclusions of the continuous dependence theorem.

4. Open this website with your browser (java-enabled):

`http://asterix.jci.tju.edu/java/lorenz.html`

Observe what happens for a couple of minutes (it is a little "video"). Write a paragraph describing what you observed.