642:515 ODE Problem Set 4

1. Let $A \in \mathbb{R}^{n\times n}$. We say that the flow of $x' = Ax$ is *positive* if $\mathbb{R}^n_+$, the nonnegative orthant of $\mathbb{R}^n$ (the set of all vectors $\xi \in \mathbb{R}^n$ such that $\xi_i \geq 0$ for all $i = 1, \dots, n$) is forward-invariant, i.e., if every solution with $x(0) \in \mathbb{R}^n_+$ satisfies $x(t) \in \mathbb{R}^n_+$ for all $t \geq 0$. Such flows are of interest in applications, since often the state variables x represent physical quantities (concentrations of chemicals, populations, etc) which cannot be negative. Positivity does not mean that all entries of A are positive (example in one dimension: the positive flow $x' = -x$ has $A = (-1)$). Prove that the following three properties are equivalent, for any given A:

(a) The flow is positive.

(b) Every entry of e^{tA} is nonnegative, for all $t \geq 0$.

(c) Every off-diagonal entry of A is nonnegative.

2. Let $A \in \mathbb{R}^{n\times n}$. We say that the flow of $x' = Ax$ is *length preserving* if $|x(t)| = |x(0)|$ along all solutions. Prove that the following three properties are equivalent, for any given A:

(a) The flow is length preserving.

(b) The matrix e^{tA} is orthogonal (i.e., $e^{tA}\left(e^{tA}\right)^T = I$) for all t.

(c) The matrix A is skew-symmetric (i.e., $A + A^T = 0$).

3. Let $A \in \mathbb{R}^{n\times n}$. We say that the flow of $x' = Ax$ is *volume preserving* if, for each Lebesgue measurable set S with finite measure, the image $e^{tA}(S)$ of S under e^{tA} has the same measure as S, for all t. Prove that the following three properties are equivalent, for any given A:

(a) The flow is volume preserving.

(b) The matrix e^{tA} has determinant one, for all t.

(c) The matrix A is zero trace.

(*Hint:* Recall, from multivariable calculus. the change of variables formula for multiple integrals, and apply it to the integrals which represent volume. If you don't want to deal with arbitrary Lebesgue measurable sets, just consider bounded "regions" in $\mathbb{R}^n$ as those treated in calculus textbooks, and if you want, you may just work out the case $n = 2$ – the ideas are the same.)

4. Consider a differential equation $x' = f(x)$, where f is continuously differentiable. We suppose that solutions exist for all $t \geq 0$, for any given initial condition $x(0) = \xi$, and write $\phi_t(\xi)$ to denote such a solution. Assume further that the *divergence* of f is everywhere zero, i.e., if $\sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(x) = 0$ for all x, where f_i is the ith coordinate of f. *Show* that the flow ϕ_t is volume-preserving, i.e. $\phi_t(S)$ and S have the same measure, for all finite-measure subsets S of $\mathbb{R}^n$. (*Hint:* The same hint as in the previous problem applies. Also, recall how to compute the Jacobian $\partial\phi_t(\xi)/\partial\xi$ via the linearized equation.)

5. A *Hamiltonian system* is a system in even dimension, let us say $n = 2m$, of the following special form. There is a function $H : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, which we take to be twice continuously differentiable, such that, if we denote coordinates of $x \in \mathbb{R}^n$ in the partitioned form $x = (q, p)$, with $q \in \mathbb{R}^m$ and $p \in \mathbb{R}^m$, then the equations look like:

$$q_i' = \frac{\partial H}{\partial p_i}, \quad p_i' = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, m.$$

(Such systems arise all the time in classical mechanics, where H is obtained as a total energy; for example, the system with $m = 1$ defined by the Hamiltonian $(1/2)(y^2 + \omega^2 x^2)$ is the harmonic oscillator $x' = y$, $y' = -\omega^2 x$.) Suppose that the level sets $\{(q,p) | H(q,p) = c\}$ are compact, so that we know that the flow $\phi_t(\xi)$ is defined for all t and all initial conditions. *Show* that ϕ_t is volume-preserving.

6. Find a continuous matrix function $A(t)$ such that $\exp\left(\int_0^t A(s)\,ds\right)$ is *not* a solution of $X' = AX$.

7. This exercise shows that deciding if all solutions converge to zero for a time-varying $x' = A(t)x$ is not simply determined by eigenvalues. For any fixed number $1 < a < 2$, consider the following matrix of functions of time:

$$A(t) = \begin{pmatrix} -1 + a\cos^2 t & 1 - a\sin t\cos t \\ -1 - a\sin t\cos t & -1 + a\sin^2 t \end{pmatrix}.$$

Prove:

(a) For every fixed $t_0 \in \mathbb{R}$, all eigenvalues of $A(t_0)$ have negative real parts.

(b) The vector function $x(t) = e^{(a-1)t}\binom{\cos t}{-\sin t}$ is a solution which does not converge to the origin (it is not even bounded).

8. Consider the system $x' = A(t)x$ studied in the previous problem, which is periodic with period 2π. Answer these questions:

(a) How do you know, using the previous problem, that one characteristic multiplier must be > 1 (and hence one characteristic exponent must be positive)?

(b) Show that the product of the two characteristic multipliers is $e^{-\pi}$. Conclude that the second multiplier must be < 1.

(c) Use a numerical package for solving ODE's (e.g. Maple), in order to find a fundamental matrix and the two multipliers. Check that their product is indeed $e^{-\pi}$ (up to numerical error).

9. Show that if λ_1 and λ_2 are two different characteristic multipliers of a system $x' = A(t)x$ with A τ-periodic, and $e^{\mu_i\tau} = \lambda_i$ for $i = 1, 2$, then there are two τ-periodic functions p_1 and p_2 so that $e^{\mu_1 t}p_1(t)$ and $e^{\mu_2 t}p_2(t)$ are linearly indepedent solutions. (It was shown in class that there is always a solution of the form $e^{\mu t}p(t)$.)

10. (Optional - this may be a bit hard.) Show that if $A(t)$ is periodic with period τ and if $x' = A(t)x$ has a solution of the form $e^{\lambda t}p(t)$ with p periodic of period τ, then λ must be a characteristic exponent. (*Hint:* Change variables so that we reduce to: if $e^{\lambda t}q(t)$ with q periodic is a solution of the constant system $z' = Rz$, then λ must be the real part of an eigenvalue of R.)

11. Show that, for every two square matrices A and B of the same size:

(a) $\frac{d}{dt}\left(e^{tA}e^{tB}\right) = A + B$.

(b) $\frac{d}{dt}\left(e^{\sqrt{t}A}e^{\sqrt{t}B}e^{-\sqrt{t}A}e^{-\sqrt{t}B}\right)|_{t=0} = [A, B] = AB - BA$ (the *Lie bracket* of A and B).

12. Suppose that $B(t)$ is a skew-symmetric $n \times n$ matrix for all t, and is continuous in t. Consider the differential equation

$$X'(t) = [X(t), B(t)] = X(t)B(t) - B(t)X(t)$$

on $n \times n$ matrices. Prove that, if $Z(\cdot)$ is the solution of $Z'(t) = Z(t)B(t)$ with $Z(0) = I$, then $Z(t)$ is orthogonal and $X(t) = Z(t)^T X(0)Z(t)$ for all t. Conclude that the flow $X' = [X, B]$ is *isospectral* (eigenvalues are preserved).

13. Let $N \in \mathbb{R}^{n\times n}$ be symmetric and consider the "double bracket flow"

$$H' = [H, [H, N]]$$

on $n \times n$ matrices. Prove that if $H(0)$ is symmetric and positive definite then also $H(t)$ is symmetric and positive definite for all those $t \geq 0$ for which the solution is defined, and $H(t)$ has the same eigenvalues as $H(0)$. Conclude that the solution is in fact defined for all $t \geq 0$ (use that the norm of $H(t)$ equals its largest eigenvalue). This problem is of interest because, when N is a matrix of the form $\mathrm{diag}(\mu_1, \ldots, \mu_n)$ with $\mu_1 > \ldots > \mu_n$, all solutions converge to a diagonal matrix whose eigenvalues are those of H, sorted. This provides an "analog computer" algorithm for eigenvalue finding for symmetric matrices, and even for sorting finite sets.