642:515 ODE Problem Set 4

1. ("Reduction of degree") Consider the homogeneous equation $x' = A(t)x$ with $A(t)$ defined and continuous on some interval I, and suppose that we know m ($m < n$) linearly independent solutions $y^1, \dots, y^m$.

(a) Show that, for any $t_0 \in I$, we may renumber the components of the vectors so that the $m \times m$ matrix $\{y^i_j\}_{i,j=1}^m$ is non-singular on an open interval J containing t_0.

(b) Let $Z(t)$ be the $n \times n$ matrix defined on J whose first m columns are $y^1, \dots, y^m$ and whose last $n-m$ columns are $e^{m+1}, \dots, e^n$. Introduce a new variable $z = (z_1, \dots z_n)$ by the formula $x(t) = Z(t)z(t)$. Show that $z_{m+1}, \dots, z_n$ satisfy a homogeneous linear system in $n-m$ variables, and if this system is solved then $z_1, \dots, z_m$ may be found by integration.

(c) Work out the details for the case $n = 2$, $m = 1$, $I = \mathbb{R}$ considered in Problem Set 4, namely, for any fixed number $1 < a < 2$, we take

$$A(t) = \begin{pmatrix} -1 + a\cos^2 t & 1 - a\sin t\cos t \\ -1 - a\sin t\cos t & -1 + a\sin^2 t \end{pmatrix}.$$

and use the first solution:

$$y^1 = e^{(a-1)t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}.$$

Find a second, linearly independent solution explicitly by this method. (What is the interval J in this case?) *Note: it is a rather interesting fact that the second solution which you will most probably obtain will be independent of a.*

2. Let $D = \mathbb{R}^2 \setminus \{0\}$, and consider the autonomous system which, written in polar coordinates, is:

$$\begin{aligned} \theta' &= \sin^2\theta + (1-r)^3 \\ r' &= r(1-r). \end{aligned}$$

Sketch the orbits in D and find ω-limit sets for all types of orbits. Next, do the same for the system

$$\begin{aligned} \theta' &= \sin^2\theta + (1-r) \\ r' &= r(1-r)^3. \end{aligned}$$

3. We consider the Lotka-Volterra equations (LV) as follows:

$$\begin{aligned} x' &= (r - \gamma x - \alpha y)x \\ y' &= (-s + \beta x - \delta y)y. \end{aligned}$$

Here the constants r, s, α, and β are positive and γ and δ are non-negative. See Example 1.1 of the notes for the ecological interpretation. Next, you are asked to sketch the phase plane under different assumptions. Give what arguments you can to support your sketch but a complete proof of all its qualitative properties is not required.

(a) Suppose that $\gamma = \delta = 0$ in (LV).

1. Find and classify all equilibrium points.
2. Describe the flow on the positive x and y axes.
3. Show that the function $F(x,y) = \alpha y - r\log y + \beta x - s\log x$ is a constant of the motion. From this, show that all orbits in the first quadrant $x, y > 0$ are periodic. Sketch the portion of the phase plane in which $x, y \geq 0$.

(b) Now suppose $\gamma, \delta > 0$ in (LV).

1. Find and classify all equilibrium points.
2. Describe the flow on the positive x and y axes.
3. Try to determine whether or not limit cycles exist. Suggested methods: the library, a computer, deep thinking.

4. Let θ be a real number. Consider the autonomous system in $\mathbb{R}^2 \times \mathbb{R}^2$ given by

$$\begin{aligned} x_1' &= -2\pi y_1 \\ y_1' &= 2\pi x_1 \\ x_2' &= -2\pi\theta y_2 \\ y_2' &= 2\pi\theta x_2 \,. \end{aligned}$$

(a) Find explicitly the flow $\Phi_t : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$.

(b) If $a, b \geq 0$, let $C_{ab} \equiv \{(x_1, y_1, x_2, y_2) \mid x_1^2 + y_1^2 = a^2, x_2^2 + y_2^2 = b^2\}$. Show that C_{ab} is an invariant set for this flow. (Note that C_{ab} is a torus if $a, b > 0$).

(c) Suppose that θ is irrational. Show that if $x(t)$ is a solution with $x(t) \in C_{ab}$ for some t and for $a, b > 0$ then the orbit of $x(t)$ is a proper subset of C_{ab}, but $\Omega[x] = C_{ab}$.

(d) Describe the orbits and ω-limit sets when θ is rational.

5. Use the Poincaré-Bendixson theorem to show that the system

$$\begin{aligned} x' &= x - y - x^5 \\ y' &= x + y - y^3 \end{aligned}$$

has a limit cycle. Hint: consider the flow on the boundary of an appropriate annulus $a < x^2 + y^2 < b$.

6. A *gradient system* is one of the form $x' = (\nabla V(x))'$ (prime indicates transpose), for some $V : \mathbb{R}^n \to \mathbb{R}$ (which we assume to be $\mathcal{C}^2$). Prove that a gradient system does not have periodic orbits.

7. Consider the following system in $\mathbb{R}^3$:

$$\begin{aligned} x_1' &= (x_3 - 0.7)x_1 - 3.5x_2 \\ x_2' &= 3.5x_1 + (x_3 - 0.7)x_2 \\ x_3' &= 0.6 + x_3 - 0.33x_3^3 - (x_1^2 + x_2^2)(1 + 0.025x_3) \,. \end{aligned}$$

Plot (in 3d) the solution with $x(0) = (0.1, 0.03, 0.001)$ as well as some other nearby initial conditions. What does the attractor that you see look like? (Describe in words and provide a printout. Note: it is not a limit cycle nor an equilibrium.)

8. Give an explicit example of an autonomous system in $\mathbb{R}^2$ for which some ω-limit set is disconnected (give equations for the example that is shown pictorially in the notes).

9. Show that, for this system in $\mathbb{R}^2$:

$$\begin{aligned} x' &= 2x \\ y' &= -4y + x^2 \end{aligned}$$

and letting A be its linearization at the origin, there is not $\mathcal{C}^2$ diffeomorphisn H such that $H(\Phi_1(\xi)) = e^A H(\xi)$ for all ξ near zero. That is, the Hartman-Grobman Theorem cannot be extended to a $\mathcal{C}^2$ diffeo. (Hint: it is enough to show that the Jacobian of H must vanish at 0 (why?). To see this, find a formula for $\Phi_1(\xi)$ and look at the second coordinate of the equality $H(\Phi_1(\xi)) = e^A H(\xi)$. Take derivatives with respect to x twice.)

10. Recall that we defined in class the *Lyapunov exponents* of $x' = A(t)x$ as follows: for each $v \in \mathbb{R}^n$,

$$\lambda_v := \limsup_{t \to +\infty} \frac{1}{t} \ln |X(t)v|$$

(where $X(t)$ is the fundamental matrix with $X(0) = I$). Show that, if $A(t)$ is periodic and μ is a Floquet exponent, then the real part of μ is a Lyapunov exponent, and the limsup is in fact a limit.