Review problems: Math 527, Exam 2

Problems 1–6 are from last year's second midterm exam; problem 7 is additional.

1. Consider the system $x' = 1 - xy$, $y' = x - y^3$. Determine its singular (equilibrium) points and classify each, insofar as possible, using linearization. In particular, classify each equilibrium point as stable or not stable. For each equilibrium point, determine in addition if it is a focus, node, or saddle, if you have the information to do so. If one cannot determine the type from the linearization, say so, and indicate, if possible, the alternatives. No sketch is required.

2. Consider the linear system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} -4 & 4 \\ 1 & -4 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

If A is the matrix of coefficients, $\det(A - \lambda I) = (\lambda + 2)(\lambda + 6)$.

Classify the singular point at the origin. (Is it stable or unstable, a focus, node, saddle, etc.?) Sketch the phase portrait near the origin. The sketch can be rough but it should have the right qualitative features, and, if there are lines through eigenvectors in the picture, they should point in roughly the correct directions. You do **NOT** need to compute and sketch $x-$ and $y-$ isoclines.

3. Show that phase trajectories of the system $x' = -8xy$, $y' = x^2$ lie in curves of the form $8y^2 + x^2 = c$. Sketch a phase portrait with several trajectories. Indicate all singular (equilibrium) points of the system. Indicate the direction of motion of the trajectories. Are trajectories of this system periodic or not?

4. Let $f(x) = 1 - x$ on $(0, 1)$. Consider the following series

$$S_1(x) = \frac{1}{2} + \sum_0^\infty \frac{1}{n\pi} \sin(2n\pi x)$$

$$S_2(x) = \frac{1}{2} + \sum_0^\infty \frac{1}{n^3} \cos(3nx)$$

$$S_3(x) = \sum_1^\infty \frac{2}{n\pi} \sin(n\pi x)$$

$$S_4(x) = \sum_{k=0}^\infty \frac{2}{\pi^2 (k + 1/2)^2} \cos\left(\frac{2k+1}{2}\pi x\right)$$

$$S_5(x) = \frac{1}{2} + \sum_0^\infty \frac{2(1 - \cos(n\pi))}{\pi^2 n^2} \cos(n\pi x)$$

$$S_6(x) = \sum_0^\infty \frac{2}{\pi^2} \left[\frac{\pi}{k + 1/2} + \frac{(-1)^{k+1}}{(k + 1/2)^2}\right] \sin\left(\frac{2k+1}{2}\pi x\right).$$

(a) One of these series is the ordinary Fourier series $FS(f)$ of f for the extension of f with period 1, one is the half-range cosine series $HRC(f)$, one is the quarter-range cosine series $QRC(f)$, one is the half-range sine series HRS(f), one is quarter-range sine series $QRS(f)$, and one is none of these. By looking at periodicity and series form, determine which is which.

(b) Graph the extensions of f corresponding to the half range sine series (HRS) and the quarter range sine series (QRS) series.

5. Find the half range sine series of $\cos(x)$ on $(0, \pi)$.

6. Let $\langle x, y\rangle = x_1y_1 + 3x_2y_2 + x_3y_3 + x_4y_4$ define an inner product on R^3 with norm $\|x\| = \sqrt{\langle x, x\rangle}$. Note: This is **NOT** the dot product! Let

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad , v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

The Gram-Schmidt method applied to the first two vectors in $\{v_1, v_2, v_3\}$ in that order gives

$$\hat{e}_1 = v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \hat{e}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Find the last vector $\hat{e}_3$ by the Gram-Schmidt method.
(b) Supppose w is a vector satisfying $\langle w, \hat{e}_1\rangle = 3$, $\langle w, \hat{e}_2\rangle = 2\sqrt{5}$ and $\|w\| = 6$. Find the vector T which is the best approximation to w in Span$\{\hat{e}_1, \hat{e}_2\}$ in the sense of minimizing the magnitude of the error $\|w - T\|^2$. What is the error $\|w - T\|^2$?

7. Solve the problem

$$\begin{gathered} u_{xx} = \alpha^2 u_t, \quad 0 < x < \pi, \ t > 0; \\ u_x(0, t) = 0, \quad u(\pi, t) = 0; \\ u(x, 0) = \cos x. \end{gathered}$$