section on the one hand and to learn how to use such software as well. To illustrate, let us use the *Maple* dsolve command (discussed at the end of Section 4.2) to obtain a Frobenius-type solution of the differential equation $xy'' + y = 0$ about the regular singular point $x = 0$; this was our Example 6. Enter

$$\text{dsolve}(x * \text{ diff}(y(x),\ x,\ x) + y(x) = 0,\ y(x),\ \text{type} = \text{series});$$

and return. The resulting output

$$y(x) = \_C1\, x\left(1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{144}x^3 + \frac{1}{2880}x^4 - \frac{1}{86400}x^5 + O(x^6)\right)$$
$$+\_C2\left[\ln(x)\left(-x + \frac{1}{2}x^2 - \frac{1}{12}x^3 + \frac{1}{144}x^4 - \frac{1}{2880}x^5 + O(x^6)\right)\right.$$
$$\left. + \left(1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \frac{101}{86400}x^5 + O(x^6)\right)\right]$$

is found to agree with the general solution that we generated in Example 5.

EXERCISES 4.3

1. For each equation, identify all singular points (if any), and classify each as regular or irregular. For each regular singular point use Theorem 4.3.1 to determine the minimum possible radii of convergence of the series that will result in (40) and (41) (but you need not work out those series).

(a) $y'' - x^3y' + xy = 0$
(b) $xy'' - (\cos x)y' + 5y = 0$
(c) $(x^2 - 3)y'' - y = 0$
(d) $x(x^2 + 3)y'' + y = 0$
(e) $(x + 1)^2y'' - 4y' + (x + 1)y = 0$
(f) $y'' + (\ln x)y' + 2y = 0$
(g) $(x - 1)(x + 3)^2y'' + y' + y = 0$
(h) $xy'' + (\sin x)y' - (\cos x)y = 0$
(i) $x(x^4 + 2)y'' + y = 0$
(j) $(x^4 - 1)y'' + xy' - x^2y = 0$
(k) $(x^4 - 1)^3y'' + (x^2 - 1)^2y' - y = 0$
(l) $(x^4 - 1)^3y'' - 3(x + 1)^2y' + x(x + 1)y = 0$
(m) $(xy')' - 5y = 0$
(n) $\left[x^3(x - 1)y'\right]' + 2y = 0$
(o) $2x^2y'' - xy' + 7y = 0$
(p) $xy'' + 4y' = 0$
(q) $x^2y'' - 3y = 0$
(r) $2x^2y'' + \sqrt{\pi}\, y = 0$

2. Sometimes one can change an irregular singular point to a regular singular point, by suitable change of variables, so that the Frobenius theory can be applied. The purpose of this exercise is to present such a case. We noted, in Example 3, that $y'' + \sqrt{x}\,y = 0$ $(x > 0)$ has an irregular singular point at $x = 0$, because of the $\sqrt{x}$.

(a) Show that if we change the independent variable from x to t, say, according to $\sqrt{x} = t$, then the equation on $y(x(t)) = Y(t)$ is

$$Y''(t) - \frac{1}{t}Y'(t) + 4t^3Y(t) = 0. \qquad (t > 0) \qquad (2.1)$$

(b) Show that (2.1) has a regular singular point at $t = 0$ (which point corresponds to $x = 0$).
(c) Obtain a general solution of (2.1) by the Frobenius method. (If possible, give the general term of any series obtained.) Putting $t = \sqrt{x}$ in that result, obtain the corresponding general solution of $y'' + \sqrt{x}\,y = 0$. Is that general solution for $y(x)$ of Frobenius form? Explain.
(d) Use computer software to find a general solution.

3. In each case, there is a regular singular point at the left end of the stated x interval; call that point x_0. Merely introduce a change of independent variable, from x to t, according to $x - x_0 = t$, and obtain the new differential equation on $y(x(t)) = Y(t)$. You need not solve that equation.