

Various Fourier-type expansions:

We write $f(x) \sim$ series to indicate that the given series is some Fourier-type expansion of $f(x)$.

$$\left. \begin{aligned} f(x) &\sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right]; \\ a_0 &= \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) dx, \quad a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx, \quad b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx. \end{aligned} \right\} \quad (1)$$

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}; \quad c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-in\pi x/\ell} dx. \quad (2)$$

$$\left. \begin{aligned} f(x) &\sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \quad (0 < x < L); \\ a_0 &= \frac{1}{L} \int_0^L f(x) dx, \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx. \end{aligned} \right\} \quad (3)$$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad (0 < x < L); \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx. \quad (4)$$

$$f(x) \sim \sum_{n=1,3,5,\dots} a_n \cos \frac{n\pi x}{2L} \quad (0 < x < L); \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{2L} dx. \quad (5)$$

$$f(x) \sim \sum_{n=1,3,5,\dots} b_n \sin \frac{n\pi x}{2L} \quad (0 < x < L); \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{2L} dx. \quad (6)$$

Some antiderivatives:

$$\begin{aligned} \int e^{-ax} \sin(bx) dx &= -\frac{e^{-ax}}{a^2 + b^2} (a \sin(bx) + b \cos(bx)) + C \\ \int e^{-ax} \cos(bx) dx &= \frac{e^{-ax}}{a^2 + b^2} (-a \cos(bx) + b \sin(bx)) + C \\ \int x \cos(bx) dx &= \frac{\cos(bx)}{b^2} + \frac{x \sin(bx)}{b} + C & \int x \sin(bx) dx &= \frac{\sin(bx)}{b^2} - \frac{x \cos(bx)}{b} + C \end{aligned}$$

Some trig identities:

$$\begin{aligned} \cos^2 x &= \frac{1 + \cos(2x)}{2} & \sin^2 x &= \frac{1 - \cos(2x)}{2} \\ \cos(A) \cos(B) &= (1/2) [\cos(A+B) + \cos(A-B)] \\ \sin(A) \cos(B) &= (1/2) [\sin(A+B) + \sin(A-B)] \\ \sin(A) \sin(B) &= (1/2) [\cos(A-B) - \cos(A+B)] \end{aligned}$$

Sturm-Liouville problem:

$$[p(x)y']' + q(x)y + \lambda w(x)y = 0, \quad \langle f(x), y(x) \rangle_w = \int_0^L f(x)g(x)w(x) dx$$