A. Centers

Consider a linear system

$$\mathbf{z}' = A\mathbf{z}, \qquad \text{with} \qquad \mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{4.1}$$

for which the origin of the phase plane is a center, that is, for which the eigenvalues of A are pure imaginary. Recall that if $p = a + d = \operatorname{Tr} A$ and $q = ab - cd = \det A$ then any eigenvalue λ satisfies $\lambda^2 - p\lambda + q = 0$; this equation has solutions $\lambda_\pm = (p \pm \sqrt{p^2 - 4q})/2$. The origin is a center if the eigenvalues $\lambda_\pm$ are pure imaginary, which requires that

$$\operatorname{Tr} A = p = a + d = 0, \tag{4.2a}$$

$$\det A = q = ad - bc > 0. \tag{4.2b}$$

In the remainder of this section we assume that (4.2) holds, so that A has eigenvalues $\lambda_\pm = \pm i\mu$ with $\mu = \sqrt{ad - bc}$.

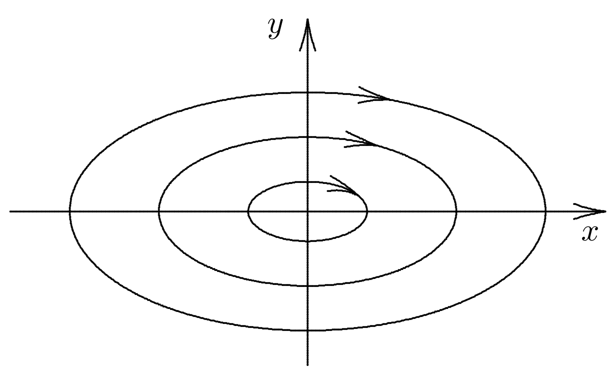
1. A special case. Consider first the special case in which $a = d = 0$. Then (4.2a) is satisfied and (4.2b) requires that c and d have opposite signs. Suppose, for example, that $b = \alpha^2 > 0$ and $c = -\beta^2 < 0$. Then the equations become $x' = \alpha^2 y$, $y' = -\beta^2 x$, so that the quantity

$$Q = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} \tag{4.3}$$

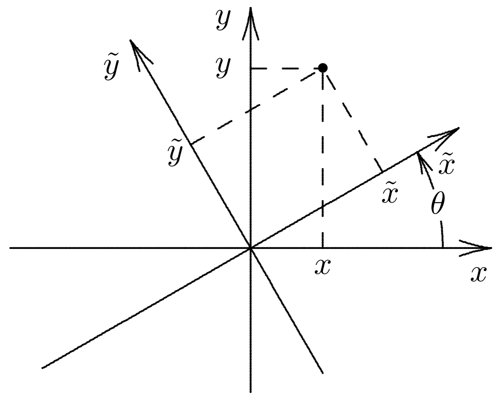
satisfies $Q' = 0$, i.e., Q is *conserved*: it is constant during the motion. The trajectories or orbits of the system are thus given by the level curves of the function Q, so that these orbits have the form

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = \gamma^2. \tag{4.4}$$

But (4.4) is just the equation of an ellipse whose axes lie along the coordinate axes; the axis of the ellipse in the x direction has length $2\alpha\gamma$ and that in the y direction has length $2\beta\gamma$. These axis lengths of course vary as γ varies, but the ratio of their lengths, α/β, is the same for all the elliptical orbits. If $b > 0$, as we have assumed above, then in the upper half plane, where $y > 0$, $x' = by > 0$ so that x increases as t increases: the trajectories circle the origin clockwise. Here is a typical picture of some trajectories in the phase plane:



2. The general case. Now consider a general system (4.1) for which A satisfies (4.2). We claim that the trajectories of the system are still ellipses, but now with axes rotated with respect to the coordinate axes. To show this, and to calculate the angle of rotation, we first rewrite the equations (4.1) in a rotated coordinate system. Let $\tilde{\mathbf{z}}$ denote the coordinates in a system in which the axes have been rotated counterclockwise through an angle θ:



Here the indicated point has coordinates $\mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix}$ in the original system and $\tilde{\mathbf{z}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}$ in the rotated system. The coordinates are related by

$$\tilde{\mathbf{z}} = R\mathbf{z}, \qquad \text{where} \qquad R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad R^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Then the differential equations become in the new coordinate system

$$\tilde{\mathbf{z}}' = \tilde{A}\tilde{\mathbf{z}}, \qquad \text{where} \qquad A \equiv \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix} = RAR^{-1}.$$

Now we want to pick θ so that in the new coordinate system the system reduces to the special case considered above, that is, so that $\tilde{a} = \tilde{d} = 0$. Using $a + d = 0$ (see (4.2a)) we find easily that

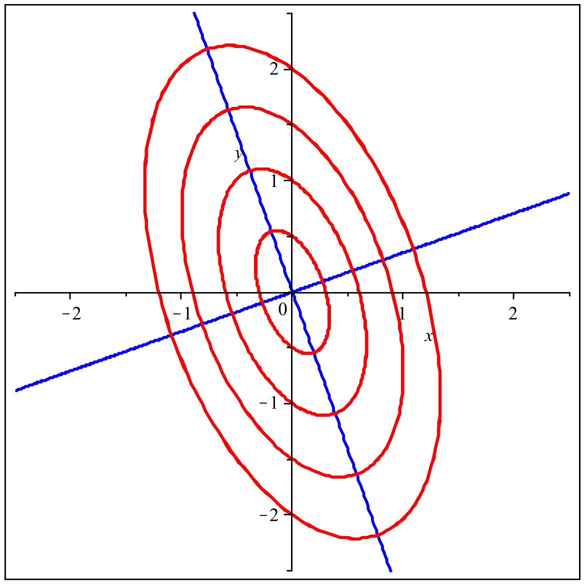
$$\tilde{a} = -\tilde{d} = \frac{a-d}{2}\cos 2\theta + \frac{b+c}{2}\sin 2\theta.$$

Thus we will have $\tilde{a} = \tilde{d} = 0$ if θ is any root of

$$\tan 2\theta = -\frac{a-d}{b+c}. \tag{4.5}$$

If θ_0 is any solution to (4.5) then $\theta_0 + \pi/2$ is also a solution, corresponding to the second axis of the ellipse (which is perpendicular to the first). Of course, $\theta_0 + k\pi/2$, $k = 0, \pm 1, \pm 2, \ldots$ are also solutions. Once one chooses a value of θ one obtains $\tilde{b}$ and $\tilde{c}$ from $\tilde{A} = RAR^{-1}$ and thus the ratio of the axes of the ellipses and the direction of motion along them, as in the special case considered above.

3. Further remarks: (a) Consider as an example the system of Greenberg, Section 7.3, equation (16). Here $a = \sqrt{8}/3$, $b = 4/3$, $c = -11/3$, and $d = -\sqrt{8}/3$, so that (4.5) becomes $\tan 2\theta = 2\sqrt{8}/7 = 0.808122$, and this indeed has solution $\theta = 19.47^\circ$ as given. However, a sketch (obtained using Maple) of the trajectories in this case shows that Greenberg's sketch (Figure 5 of Section 7.3) interchanges the major and minor axes of the ellipses:



Maple finds that $\tilde{b} = 1$ and $\tilde{c} = -4$, so that the major axis of any of these ellipses is twice as long as the minor axis.

(b) There are at least two other approaches to the problem of determining θ. For the first, note that for the general system (4.1) there will again be a conserved quantity Q whose level curves are the trajectories—in this case, the tilted ellipses. Q will have the form

$$Q(\mathbf{z}) = \mathbf{z}^T M \mathbf{z} = f\,x^2 + 2g\,xy + h\,y^2, \qquad \text{where} \quad M = \begin{bmatrix} f & g \\ g & h \end{bmatrix}.$$

Here $\mathbf{z}^T$ is the transpose of $\mathbf{z}$. Since (4.1) implies that $Q' = \mathbf{z}^T A^T M \mathbf{z} + \mathbf{z} M A \mathbf{z}$, the matrix M can be determined by solving $A^T M + MA = 0$. Now M is a symmetric matrix and hence can be diagonalized by a rotation; the rotation matrix is exactly the matrix R above. The diagonalized matrix $\tilde{M}$ must yield a conserved quantity $\tilde{Q} = \tilde{\mathbf{z}}^T \tilde{M} \tilde{\mathbf{z}}$ of the form (4.3) and the eigenvalues of M are thus proportional to the quantities $1/\alpha^2$, $1/\beta^2$ which give the lengths of the axes of the ellipses.

(c) A second alternative approach is to solve the system (4.1) and consider any specific solution $\mathbf{z}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. Then $r^2(t) = x^2(t) + y^2(t)$ will be the square of the distance from $\mathbf{z}(t)$ to the origin, and this will be a maximum when $\mathbf{z}(t)$ crosses the major axis of the ellipse and a minimum when it crosses the minor axis. Setting $d(r^2)/dt = 0$ will enable one to find the corresponding values of t and, substituting these into $\mathbf{z}(t)$, the angles and lengths of the axes.

B. Foci (spirals)

Greenberg also wishes to define an angle θ when the origin is a focus (stable or unstable) for the system (4.1). This seems a bit problematical: there are now at least two possible ways to define θ; these are inconsistent and it is not clear in what sense either is natural.

To follow the approach which Greenberg takes we first solve the equations; if the eigenvalues of A are $\nu \pm i\mu$ then one solution may be put in the form $x(t) = e^{\nu t}\cos(\mu t)$, $y(t) = Ce^{\nu t}\cos(\mu t - \phi)$ (and all other solutions will then have the form $\hat{x}(t) = Kx(t - t_0)$, $\hat{y}(t) = Ky(t - t_0)$). Now we obtain new trajectories, say $\xi(t), \eta(t)$, by simply dropping the factor $e^{\nu t}$ from these: $\xi(t) = \cos(\mu t)$, $\eta(t) = C\cos(\mu t - \phi)$. These trajectories will be elliptical and we may thus associate with them a rotation angle θ, according to the analysis in **A** above; this is the angle which Greenberg uses in analyzing the equations (18) of Section 7.3 (see his Figure 6). The new trajectories will in fact satisfy the equations

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix}' = \left(A - \frac{a+d}{2} I \right) \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{4.6}$$

so that the analysis can start from (4.6). Note that the coefficient matrix $A - (a+d)I/2$ in (4.6) is simply $A - (1/2)\,\mathrm{Tr}\,A$, that is, we have converted A into a matrix of trace zero in the simplest posible way.

One might also try for the foci the approach of **3**(c) above: to find the value of t at which the distance from the trajectory to the origin is a local maximum or local minimum (there will be no absolute max or min). However, this will give a different answer from tthe one obtained by analyzing (4.6).