11/12/09

(RECALL)

STURM–LIOUVILLE THEORY

(ODE) $(p(x)y')' + q(x)y + \lambda w(x) y = 0$, $a < x < b$

p, p', q, w are all continuous $a \le x \le b$

and $p(x)$, $w(x)$ are strictly positive ie > 0.

} Regular S-L Problem.

If some of these conditions don't hold, $\Rightarrow$ SINGULAR S-L Problem.

Boundary cond^n: $\alpha y(a) + \beta y'(a) = 0$ & $\gamma y(b) + \delta y'(b) = 0$

. SEPARATED Boundary Cond^n (two ends are separate. what's happening on one end doesn't related to on the other)

We may have Boundary Conditions which are not separated.

Ex: Periodic BC: $y(a) = y(b)$ $y'(a) = y'(b)$

(SECTION 17.8)

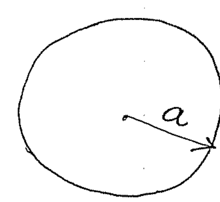
SINGULAR S-L PROBLEMS or NON-SEPARATED BOUND. COND^N.

For singular S-L problems or non-separated Boundary Conditions, part of our theorem can fail.

Example: Heat Conduction in a circular disk. (See Notes on this on webpage)

disc: radius 'a'

$u(x, y, t)$: temperature.

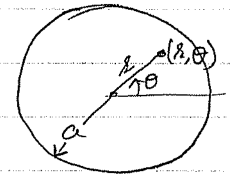


Heat equation in 2-D:

$$u_t = \alpha^2 (u_{xx} + u_{yy}) = \alpha^2 \Delta u = \alpha^2 \vec{\nabla}^2 u$$

(as Laplace eq^n form)

Go to polar co-ordinates, $u(r,\theta,t)$



PDE: $$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}}$$

To get this, start with $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$

[Change $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \to \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$ using chain rule.]

IC $\to$ $u(r,\theta,0) = f(r,\theta)$ $\to$ (some function of r,θ)

BC $\to$ $u(a,\theta,t) = 0 \quad \forall\, \theta, t>0$ (Holding the edge of disc at zero temp.)

(Now solve by Separation of Variables)

1) Look for product solutions of PDE.
2) Impose BC $\to$ eigenvalues
3) Satisfy IC using a linear combination of solutions in 2).

Look for solutions $u(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t)$

(Plugging it in the PDE)

$\Rightarrow \frac{1}{r}(rR')'\,\Theta T + \frac{1}{r^2}\Theta'' RT = \frac{1}{\alpha^2} R\,\Theta\, T'$ divide this eq^n by $R\Theta T$

$\Rightarrow \underbrace{\frac{1}{R}\cdot\frac{1}{r}(rR')' + \frac{1}{r^2}\cdot\frac{1}{\Theta}\Theta''}_{\text{depend only on } R,\Theta} = \underbrace{\frac{1}{\alpha^2}\frac{T'}{T}}_{\text{only on } T} = -\lambda$

equality $\Rightarrow$ only possible when both sides are constant ($-\lambda$: say).

(RHS)

Sol^n for T $\Rightarrow$ $T' = -\lambda\alpha^2 T \quad \Rightarrow \quad T(t) = e^{-\lambda\alpha^2 T}$ — (1).

$\to$ (Can put a constant here BUT we add the constants while doing superposition of sol^n. $\Rightarrow$ It's OK even if we putt constt. now)

(LHS)

$\Rightarrow -\frac{\Theta''}{r^2\Theta} = \lambda + \frac{1}{rR}(rR')'$

$\hookrightarrow$ Multiply w/ r^2. $\Rightarrow -\frac{\Theta''}{\Theta} = \lambda r^2 + \frac{r}{R}(rR')'$ $\to$ again this equality will hold iff both sides are constts.

$\Rightarrow -\frac{\Theta''}{\Theta} = \lambda r^2 + \frac{r}{R}(rR')' = \text{constt.} = \mu.$

$\Rightarrow \Theta'' + \mu\Theta = 0$ (ODE in θ) $\quad 0 \leqslant \theta \leqslant 2\pi$ BUT BC?

$\hookrightarrow$ equal at $\theta = 0, 2\pi$ as physically they are the same point. $\leftarrow$ same for slope. $\Leftarrow$ periodic Boundary Cond^n.

$\Theta(0) = \Theta(2\pi) \qquad \Theta'(0) = \Theta(2\pi)$

$\longrightarrow$ THIS IS A S-L PROBLEM W/ PERIODIC BC!

(Solve) for Θ

$\underline{\mu < 0}$: Not possible $\qquad \mu = -k^2$

$\Theta = A\cosh k\theta + B\sinh k\theta \qquad \Theta' = Ak\sinh k\theta + Bk\cosh k\theta$

$\Theta(0) = \Theta(2\pi) \Rightarrow A = A\cosh 2\pi k + B\sinh 2\pi k$

$\Theta'(0) = \Theta'(2\pi) \Rightarrow kB = kA\sinh 2\pi k + kB\cosh 2\pi k$] we know $k \neq 0$.

$\hookrightarrow B = A\sinh 2\pi k + B\cosh 2\pi k$

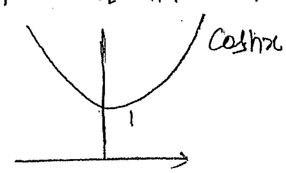
$\Rightarrow (1-\cosh 2\pi k)A - (\sinh 2\pi k)B = 0$

$(-\sinh 2\pi k)A + (1-\cosh 2\pi k)B = 0$] $\Rightarrow$ set of homogenous, linear equations

determinant $\underline{\text{has}}$ to be zero for a non-zero A, B $\qquad$ ($\Rightarrow$ only possible sol^n: $A=B=0$. visually)

$$\begin{vmatrix} 1-\cosh 2\pi k & \sinh 2\pi k \\ -\sinh 2\pi k & 1-\cosh 2\pi k \end{vmatrix} = 1 + \cosh^2 2\pi k - 2\cosh 2\pi k - \sinh^2 2\pi k$$

$(\cosh^2 x - \sinh^2 x = 1)$



$= 2(1-\cosh 2\pi k) = 0 \Rightarrow \cosh 2\pi k = 1$

which is possible for $k^2 = 0$

$\underline{\text{BUT}}$ $\cancel{\mu = -k^2 \;\&\; k > 0}$

$\mu = \underline{-k^2 < 0}$ $\quad \therefore k = 0$ is not possible in this case.

$\underline{\mu > 0}$: $\mu = k^2$

$\Theta = A\cos k\theta + B\sin k\theta$ (Solving like above) $\cos 2\pi k = 1$

$\Rightarrow k = n = 1, 2, 3 \ldots$

$\Rightarrow$ Eigenvalues: $\mu_n = n^2$; $n = 1, 2, 3$

(Periodic Boundary Cond^n still same, gives) $(1-\cos 2\pi k)A - (\sin 2\pi k)B = 0$

$(\sin 2\pi k)A + (1-\cos 2\pi k)B = 0$

for $k = n$ $\Rightarrow 0\cdot A - 0\cdot B = 0$; $0A + 0\cdot B = 0$

which is true for all A, B.

S-L Theorem said one eigen fn for one eigen value w/c doesn't hold now!!

$\Rightarrow$ For $\mu_n = n^2$, we have $\underline{\text{two eigenfunctions}}$!! $\quad \Theta_{n,1} = \sin n\theta$, $\Theta_{n,2} = \cos n\theta$ — (2)

$\underline{\mu = 0}$: $\Theta = A + B\theta$

$\Theta(0) = \Theta(2\pi) \Rightarrow B = 0 \qquad \Theta'(0) = \Theta'(2\pi) = 0 \Rightarrow A$ can be anything

[constant sol^n satisfies both BC/IC $\therefore$ sol^n is any constt. say $= 1$]

$\mu_0 = 0 \quad \Theta_0 = 1$ — (3)

(Solve for R) $r(rR')' - n^2 R + \lambda r^2 R = 0$ (from eq^n $= \mu \equiv n^2$ const.) $0 \le r \le a$ (divide by r)

$\Rightarrow (rR')' - \frac{n^2}{r} R + \lambda r R = 0$ S-L problem: $(py')' + qy + \lambda w y = 0$.

$\Rightarrow p(r) = r \quad q(r) = -\frac{n^2}{r} \quad w(r) = r$

BUT $p(0) = w(0) = 0$ & $q(r) \to \infty$ as $r \to 0$ $\Rightarrow$ SINGULAR S-L Problem

Cond^n of S-L Th. FAIL: S-L Theorem's conditions that p, w are strictly positive fails. Also, q is not continuous @ 0 w/c is in range $0 \le r \le a$.

BC: $R(a) = 0$ (from $u(a, \theta, t) = 0$)

$r = 0$ is a pt.!! $\Rightarrow$ No real BC as before at $r = 0$.

As we can see from above p, w vanish at $r = 0$ & $q \to \infty$

$\Rightarrow r = 0$ is a Singular Point.

Require R be finite as $r \to 0$ [since we don't want Temp., $u \to \infty$ $\Rightarrow$ for finite $u \to$ need finite R]

$r(rR')' - n^2 R + \lambda r^2 R = 0 \Rightarrow r^2 R'' + rR' + (\lambda r^2 - n^2) R = 0$

Almost Bessel's equation!

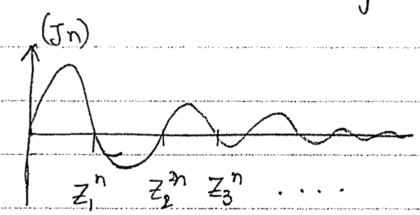
$\Rightarrow$ New Variable : $s = \sqrt{\lambda}\, r$ (λ is positive)

$\Rightarrow s^2 \frac{d^2 R}{ds^2} + s \frac{dR}{ds} + (s^2 - n^2) R = 0 \Rightarrow$ Bessel's equation of order n!

(we know from before) $R(r) = A J_n(\sqrt{\lambda}\, r) + B Y_n(\sqrt{\lambda}\, r)$ (see Lecture on 09/17/09)

Now, $Y_n \to \infty$ as $r \to 0$ BUT $R(r)$ has to be finite $\Rightarrow B = 0$.

$\Rightarrow R(r) = J_n(\sqrt{\lambda}\, r)$



BC: $R(a) = 0$

z_k^n : k^{th} positive zero of J_n

$\Rightarrow$ MUST Have $\sqrt{\lambda}\, a = z_k^n$ for some k for $J_n(\sqrt{\lambda}\, a) = 0$!

$$\Rightarrow \lambda_k^n = \left(\frac{z_k^n}{a}\right)^2$$

($n \equiv$ which Bessel's fn we're looking at, $k \equiv$ w/c eigenvalue of that Bessel fn)

Eigenfunction: $\phi_k^n(r) = J_n\left(\frac{z_k^n}{a} r\right)$

Product solutions:

$n=0$ $\quad \phi_k^0(r)\, e^{-\alpha^2 \lambda_k^0 t}$ $\quad$ (for $n=0$ $\theta=$ const. value) $K=1,2,3\ldots$

(we've a family of eigenfunctions for $n=0$!)

$n>0$: $\quad \phi_K^n(r).(\cos n\theta)\, e^{-\alpha^2\lambda_K^n t}$ & $\phi_K^n(r).(\sin n\theta)\, e^{-\alpha^2\lambda_K^n t}$ $\quad K=1,2,3\ldots$

General Solution:

$$\boxed{u(r,\theta,t) = \sum_{K=1}^{\infty} C_K^0\, \phi_K^0(r)\, e^{-\alpha^2\lambda_K^0 t} + \sum_{n=1}^{\infty}\sum_{K=1}^{\infty} \phi_K^n \left(C_K^n \cos n\theta + d_K^n \sin n\theta\right) e^{-\alpha^2\lambda_K^n t}}$$

Initial condition: $\quad u(r,\theta,0) = \sum_{K=1}^{\infty} C_K^0\, \phi_K^0(r) + \sum_{n=1}^{\infty}\sum_{K=1}^{\infty} \phi_K^n \left(C_K^n \cos n\theta + d_K^n \sin n\theta\right)$