Methods of Applied Mathematics, 642:527

Guide to Prerequisites

I. Prerequisites and entrance into Math 527.

The mathematical prerequisites for 642:527, Methods of Applied Mathematics, are the calculus of single variable and multivariable functions, elementary differential equations, and some linear algebra. The calculus and differential equations needed are what the student will have learned from his or her undergraduate courses. Improper integrals, power series, solution of first order ordinary differential equations by separation of variables, and solutions of linear, constant coefficient, second order differential equations are especially important and heavily applied. From multivariable calculus, the student should know how to compute and apply partial derivatives. The student should also know how to multiply matrices and to find eigenvalues and eigenvectors of square matrices; however, it is not necessary to have had an undergraduate course in linear algebra.

The majority of students entering Math 527 are well prepared for the course on the basis of their previous studies. However, those who have been out of school for a while, or have a weak mathematical background, may not be sufficiently prepared. To help students enroll in the course that is right for them, a short and basic entrance quiz will be given in the first Math 527 class. Students who do poorly will be advised as to alternate placement. The quiz will cover the following topics: improper integrals, partial derivatives, solving first and second order linear differential equations, solving first order separable equations, and finding eigenvalues and eigenvectors of a matrix. Some of these topics, and some others, are reviewed in the next section. Multiple integrals and partial derivatives are not reviewed, but here are a few exercises on these topics:

Exercise 1. (a) Which of the following improper integrals converge, and which diverge? Justify your answers.

$$\text{(i)} \quad \int_0^\infty \frac{t}{2t^2+1}\,dt, \qquad \text{(ii)} \quad \int_0^\infty \frac{\sin t}{t^4+3}\,dt \qquad \text{(iii)} \quad \int_0^\infty t^2 e^{-t}\,dt$$

(b) Consider the function $f(x,y,z) = \sin(x^2y) + 3y\ln z^3 - 2xye^{x^2+z^2}$. Find the partial derivatives f_x, f_y, f_z, and f_{yz}.

II. A review of selected topics.

The brief review of selected prerequisites in this section is intended to help the student prepare for Math 527. All these topics should be familiar from his or her previous courses, and all are covered in the course text, *Advanced Engineering Mathematics*, by Greenberg, in various places and in greater detail.

A. Infinite series, power series, and Taylor series. The principal concern of Math 527 is solving ordinary and partial differential equations as explicitly as possible. It turns out that simple combinations of the elementary functions learned in calculus—the algebraic, exponential, logarithmic and trigonometric functions—are usually not adequate to this task. However, infinite series of elementary functions, such as power series, which are infinite series of polynomial functions, or Fourier series, which are infinite series of trigonometric functions, do often work. The application of infinite series to differential equations is a major theme of Math 527, and so it is important to understand these series clearly. Experience shows that many students entering Math 527 need to review infinite sequences and series. While class time will be devoted to this, it is strongly advised that the student study the brief review that follows, which focuses on the most essential points.

A.i. *Infinite series: definitions.* Let $\{a_n\} = \{a_1, a_2, \ldots\}$ be a sequence of real numbers. The expression

$$\sum_{n=1}^{\infty} a_n, \qquad \text{or, written out,} \quad a_1 + a_2 + a_2 + \cdots,$$

is called an *infinite series.* If $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$ exists and is finite, we say that the infinite series *converges*, and we identify the value of the infinite series with this limit:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n \qquad \text{if the limit exists and is finite.}$$

If this limit does not exist or is infinite, the infinite series is said to *diverge.* The sums $\sum_{n=1}^{N} a_n$, where N is finite, are called *partial sums* of the infinite series.

The infinite series $\sum_{n=1}^{\infty} a_n$ is said to *converge absolutely* if $\sum_{n=1}^{\infty} |a_n|$ converges. If an infinite series converges absolutely, then it converges. A series which converges, but not absolutely, is said to be *conditionally convergent.*

Warning. An elementary mistake is to confuse convergence of an infinite series with convergence of the sequence of terms in the infinite sum. Keep clear the difference between $\lim_{n\to\infty} a_n$ and $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$; only the latter, if it exists, represents the value of the infinite series.

A.ii. *Geometric Series.* The most important example of an infinite series is the *geometric series* $\sum_0^\infty r^n$. We have

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \dfrac{1}{1-r} & \text{if } |r| < 1. \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases} \tag{1}$$

For $r \neq 1$, this is easy to see from the formula, $\sum_{n=0}^{N} r^n = \dfrac{1 - r^{N+1}}{1-r}$, for the partial sums of the geometric series, because

$$\lim_{N\to\infty} \frac{1-r^{N+1}}{1-r} = \frac{1}{1-r}\left[1 - \lim_{N\to\infty} r^{N+1}\right] = \begin{cases} \dfrac{1}{1-r}, & \text{if } |r|<1; \\ \text{diverges}, & \text{if } |r|>1 \text{ or } r=-1. \end{cases}$$

Note that if $r=-1$ then $\lim_{N\to\infty} r^{N+1}$ does not exist because the values r^{N+1} oscillate from 1 to -1 as N changes. If $r=1$, $\sum_{n=0}^{N} r^n = N+1$, which grows to infinity as $N\to\infty$, implying divergence.

A.iii. *Convergence and divergence.* The elementary theory of infinite series is concerned mostly with techniques for determining convergence or divergence.

The most basic fact is this: if $\sum_1^\infty a_n$ converges, then it must be true that $\lim_{n\to\infty} a_n = 0$. As a consequence we obtain the *divergence test:* if $\lim_{n\to\infty} a_n \neq 0$, or if this limit does not exist, then $\sum_1^\infty a_n$ must diverge.

For example, the divergence test implies $\sum_1^\infty n/(n+1)$ and $\sum_1^\infty (-1)^n$ both diverge, since, in the first case, $\lim_{n\to\infty} n/(n+1) = 1$, and, in the second, $\lim_{n\to\infty} (-1)^n$ does not exist. The divergence test also shows that the geometric series diverges whenever $|r|\geq 1$.

The divergence test is an easy first test. However, the converse of the divergence test does not hold: that is, $\lim_{n\to\infty} a_n = 0$ **does not imply** that $\sum_1^\infty a_n$ converges. The *harmonic series* $\sum_1^\infty n^{-1}$ is an example; although $\lim_{n\to\infty} n^{-1} = 0$, the harmonic series diverges. Therefore, when $\lim_{n\to\infty} a_n = 0$ one must resort to more refined tests to discriminate between convergence and divergence. These tests all involve comparison of the infinite series, either to an improper integral or to an infinite series with known convergence properties. For Math 527, the most important test will be *ratio test.* This states that:

- if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_1^\infty a_n$ converges absolutely;
- if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum_1^\infty a_n$ diverges.

When $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, the infinite series may either converge or diverge and then a different test is required to distinguish between the two possibilities.

The ratio test is best understood intuitively as a test that compares $\sum_1^\infty a_n$ to a geometric series. Let $r = \lim_{n\to\infty} a_{n+1}/a_n$. Heuristically, this says that for all large n, $a_{n+1}\approx a_n r$. Fixing a suitably large m, we get that $a_{m+1}\approx a_m r$, $a_{m+2}\approx a_{m+1} r \approx a_m r^2$, and, continuing in this manner, $a_{m+k}\approx a_m r^k$; thus, for large m, the terms a_m of the series look approximately like those of a geometric series, and accordingly the series converges if $|r|<1$ and diverges if $|r| > 1$. This is not a rigorous argument, but it is the idea behind a rigorous proof.

Here are two examples. The infinite series $\displaystyle\sum_1^\infty \frac{1}{n!}$ converges, because in this case, $a_{n+1}/a_n = 1/(n+1)$, which tends to 0 as $n \to \infty$. Likewise, $\displaystyle\sum_1^\infty \frac{n}{2^n}$ converges, because in this case

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

A.iv. p-series: The infinite series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ is called a p-series. It converges if $p > 1$ and it diverges if $p \leq 1$. In particular, as mentioned above, the harmonic series $\displaystyle\sum_{n=1}^\infty \frac{1}{n}$ diverges.

Convergence and divergence of p-series are usually proved by using the *integral comparison test,* which tests convergence by comparing the infinite series to an appropriate improper integral. Note that the ratio test does not work for p-series, because if $a_n = 1/n^p$, as in the p-series, $\lim_{n\to\infty} a_{n+1}/a_n = \lim_{n\to\infty}(n/(n+1))^p = 1$.

Exercise 2. Determine if the series converge or diverge: (a) $\displaystyle\sum_1^\infty \frac{3^n}{n!}$; (b) $\displaystyle\sum_1^\infty \frac{n^n}{n!}$.

A.v. *Power series.* A *power series* is an expression of the form

$$\sum_{n=0}^\infty c_n(x-x_0)^n. \tag{2}$$

In this series, x_0 and $c_0, c_1, \ldots$ are real numbers and x is a variable. The numbers $c_0, c_1, \ldots$ are called the *coefficients* of the power series, and one says that the power series is *centered at* x_0. The first term $c_0(x-x_0)^0$ is always taken to be equal to c_0. In a sense, infinite series are generalizations of polynomials. In fact, polynomials are special cases of power series in which only a finite number of terms are nonzero.

Power series can be used to define functions. If we define a function f by the rule

$$f(x) = \sum_0^\infty c_n(x-x_0)^n,$$

we mean that $f(x)$ assigns to x the value of the infinite series at x, *if the infinite series converges*; otherwise, if the infinite series diverges, x is outside the domain of f and $f(x)$ is left undefined.

Example. Consider $f(x) = \displaystyle\sum_{n=0}^\infty (x-1)^n$. Taking $x = 3/2$ and using the formula for the sum of a geometric series $f(3/2) = \sum_{n=0}^\infty (3/2-1)^n = \sum_{n=0}^\infty (1/2)^n = 2$. In fact, the geometric series formula implies

$$\sum_{n=0}^{\infty}(x-1)^n = \begin{cases} \frac{1}{1-(x-1)} = \frac{1}{2-x} & \text{if } |x-1| < 1. \\ \text{diverges} & \text{if } |x-1| \geq 1. \end{cases} \tag{3}$$

Hence the domain of f is the interval $0 < x < 2$ and $f(x)$ equals $1/(2-x)$ on this interval:

$$\frac{1}{2-x} = \sum_{n=0}^{\infty}(x-1)^n, \quad \text{for } 0 < x < 2. \tag{4}$$

Notice that in this equation we are not asserting that $1/(2-x)$ and $\sum_{n=0}^{\infty}(x-1)^n$ are identical functions, only that they coincide on $(0, 2)$, which happens to be the interval on which the power series converges. In working with power series, one must be attentive to intervals of convergence in this way.

The important things to know about power series are: (1) given a power series, how to find the set of values of x at which it converges; (2) how to differentiate, integrate, and algebraically manipulate power series on their domains of convergence; (3) how to find power series representations (Taylor series) of a given function f.

(1) Domains of convergence of a power series. The basic fact is this: given a power series in the form (2) there is an R, where $0 \leq R \leq \infty$, such that

(a) the power series converges if $|x - x_0| < R$, that is, on $(x_0 - R, x_0 + R)$, and

(b) it diverges if $|x - x_0| > R$, that is, on $(x_0 - R, x_0 + R)$.

The number R is called the *radius of convergence* of the power series. In the special case $R = 0$, the series converges only if $x = x_0$, (convergence at $x = x_0$ is always true because then all terms in the series are zero, except possibly the first term c_0). In the case $R = \infty$, the power series converges for all x. When $0 < R < \infty$, the power series may diverge or converge at the endpoints $x_0 - R$ and $x_0 + R$.

The radius of convergence can often be determined in practice by applying the ratio test.

Example. $\sum_{0}^{\infty} \sqrt{n}\frac{x^n}{2^n}$. According to the ratio test, this series will converge if

$$\lim_{n\to\infty} \frac{|\sqrt{n+1}x^{n+1}/2^{n+1}|}{|\sqrt{n}x^n/2^n|} < 1,$$

and will diverge if this limit is > 1. Since

$$\lim_{n\to\infty} \frac{|\sqrt{n+1}x^{n+1}/2^{n+1}|}{|\sqrt{n}x^n/2^n|} = \lim_{n\to\infty} \sqrt{\frac{n+1}{n}}\frac{|x|}{2} = \frac{|x|}{2},$$

it follows that the series converges if $|x|/2 < 1$, or , equivalently, if $|x| < 2$, and diverges if $|x| > 2$; thus the radius of convergence in this case is $R = 2$.

Example. The series $\sum_0^\infty n3^n x^2$ converges if

$$1 > \lim_{n\to\infty} \frac{|(n+1)3^{n+1}x^{2(n+1)}|}{|n3^n x^2|} = 3x^2 \lim_{n\to\infty} \frac{n+1}{n} = 3x^2,$$

and diverges if this limit is > 1. Thus the radius of convergence is determined by the inequality $3x^2 < 1$, or equivalently, $|x| < \sqrt{3}$, and hence $R = \sqrt{3}$.

For the series $\sum_0^\infty c_n(x - x_0)^n$, the ratio test requires

$$1 > \lim_{n\to\infty} \frac{|c_{n+1}(x - x_0)^{n+1}|}{|c_n(x - x_0)^n|} = |x - x_0| \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|.$$

It follows that

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{c_{n+1}}{a_n}\right|} = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

is a general formula for R, if the limit on the right hand side exists or equals ∞. (In practice the limit does not always exist—for instance, it does not exist for the last example $\sum_0^\infty n3^n x^2$ since $c_n = 0$ for odd n— so it is always safer to use the ratio rule directly.)

(2) Calculus and algebra of power series. The rule of thumb is: *on their intervals of convergence, infinite series can be treated as if they were polynomials.* This means:

(a) If a power series $\sum_0^\infty c_n(x-x_0)^n$ converges on (x_0-R, x_0+R), then it is differentiable on $(x_0 - R, x_0 + R)$ and its derivative is obtained by term-by-term differentiation:

$$\frac{d}{dx}\left[\sum_0^\infty c_n(x - x_0)^n\right] = \sum_{n=0}^\infty \frac{d}{dx} c_n(x-x_0)^n = \sum_{n=1}^\infty n c_n(x-x_0)^{n-1}, \quad x_0-R < x < x_0+R. \tag{5}$$

By repeating this procedure, one finds that power series have derivatives of all orders on their intervals of convergence and

$$\frac{d^k}{dx^k}\left[\sum_0^\infty c_n(x - x_0)^n\right] = \sum_{n=k}^\infty n(n-1)\cdots(n-k+1)c_n(x-x_0)^{n-k}, \qquad x_0-R < x < x_0+R. \tag{6}$$

(b) Power series can be integrated term-by-term on their interval of convergence. If $\sum_0^\infty c_n(x - x_0)^n$ converges on $(x_0 - R, x_0 + R)$, then

$$\int_{x_0}^x \sum_{n=0}^\infty c_n(u-x_0)^n\, du = \sum_{n=0}^\infty \int_{x_0}^x c_n(u-x_0)^n\, du \quad = \quad \sum_{n=0}^\infty \frac{c_n}{n+1}(x - x_0)^{n+1}, \tag{7}$$

$$x_0 - R < x < x_0 + R.$$

(c) Different power series centered at the same x_0 and all converging for $|x - x_0| < R$ may be added, multiplied, and divided as if they were polynomials and the resulting power series converge for $|x - x_0| < R$, except in the case of division, in which they converge so long as the denominator is not zero.

(3) Representation of functions by power series; Taylor series. Let f be a given function and let x_0 be given. We are interested, if possible, in representing f in an interval about x_0 by a convergent power series:

$$f(x) = \sum_0^\infty c_n(x - x_0)^n, \qquad x_0 - R < x < x_0 + R \tag{8}$$

By evaluating (6) at x_0 for each k, we find $c_k = \dfrac{f^{(k)}(x_0)}{k!}$, where $f^{(k)}$ denotes the derivative of f of order k. Hence the power series representation of f in (8) must have the form

$$f(x) = \sum_0^\infty \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{9}$$

This power series is called the *Taylor series* of f centered at x_0. The Taylor series of f at $x_0 = 0$ is sometimes called the *Maclaurin series* of f. If the Taylor series of f at x_0 converges to f in some interval centered at x_0 of radius $R > 0$, then f is said to be *analytic* at x_0.

The elementary special functions that are studied in undergraduate calculus are analytic at most points. The exponential and trigonometric functions, as well as the polynomials, are analytic at all points. Rational functions, that is, ratios of polynomials, are analytic at all points at which they are defined. Fractional powers x^p, where p is not an integer, are analytic everywhere they are defined except at the origin. If f is analytic at x_0 and g is analytic at $f(x_0)$, then $f(g(x))$ will be analytic at x_0. Examples: e^x is analytic at every x. On the other hand, $\sqrt{x}$ is analytic at every $x > 0$, but is not analytic at 0. $e^{\sqrt{x}}$ will be analytic at every $x > 0$, but not at 0.

The following Taylor series, given with intervals of convergence, are basic and should be known from memory:

(a) $$\frac{1}{1-x} = \sum_{n=0}^\infty x^n, \qquad |x| < 1.$$

(b) $$e^x = \sum_{n=0}^\infty \frac{x^n}{n!}, \qquad |x| < \infty.$$

(c) $$\cos x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{(2n)!}, \qquad |x| < \infty.$$

(d) $$\sin x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty.$$

Finding Taylor series. In the very simplest cases one can use (9). However, this is often impractical or clumsy, and it may be simpler to exploit simple known power series using substitution and possibly differentiation or integration.

Example. Find the Taylor series of $\dfrac{1}{x+2}$ and its radius of convergence about the point $x_0 = 1$.

Solution. We are looking for a Taylor series in the form $\sum_{n=0}^{\infty} c_n(x-1)^n$, so we write

$$\frac{1}{x+2} = \frac{1}{3+(x-1)} = \frac{1}{3}\left[\frac{1}{1-(-(x-1)/3)}\right].$$

Substituting $-(x-1)/3$ for x in the geometric power series formula (a) above,

$$\frac{1}{x+2} = \frac{1}{3}\sum_{n=0}^{\infty}\left[-\frac{x-1}{3}\right]^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-1)^n, \qquad \left|\frac{x-1}{3}\right| < 1.$$

This is the Taylor series and its radius of convergence is $R = 3$.

Exercise 3. (a) Find the Taylor centered at $x = 3$ for $f(x) = 1/(2+x)$.

(b) Find the Taylor series centered at $x = 0$ for $f(x) = e^{x^2}$. (Try solving this one first by direct use of (9) and see the kind of problem you run into! Then use substitution.)

(c) Find the Taylor series centered at $x = 0$ for $f(x) = \ln(1-x)$ in two ways: first, from (9), then by integrating the geometric series for $1/(1-x)$.

B. Complex exponentials. It is assumed that the student knows the definition of a complex number and the rules for working with complex numbers algebraically, and understands complex roots of a quadratic polynomial.

The elementary functions defined in calculus courses all have extensions to functions of a complex variable. Exponentials of complex numbers are particularly useful. They are defined by taking the Taylor series presented in (b) of the previous page as a definition. For a complex number $z = x + iy$, where x and y are real numbers, define

$$e^z = e^{x+iy} = \sum_{n=0}^{\infty}\frac{(x+iy)^n}{n!}. \tag{10}$$

This definition automatically guarantees the usual rule for multiplying exponentials by adding exponents: $e^{z_1}e^{z_2} = e^{z_1+z_2}$ (to see this requires a bit of nasty algebra). The student may check by direct calculation and the Taylor series (c) and (d) on the previous page, that $e^{iy} = \cos y + i\sin y$. Thus,

$$e^{x+iy} = e^x(\cos y + i\sin y). \tag{11}$$

Most importantly for differential equation applications, the formula $(d/dt)e^{\alpha t} = \alpha e^{\alpha t}$ of elementary calculus generalizes to complex α:

$$\frac{d}{dt}e^{(a+bi)t} = \frac{d}{dt}\left[e^{at}\cos(bt) + i\frac{d}{dt}e^{at}\sin(bt)\right] = (a+bi)e^{(a+bi)t}. \tag{12}$$

Exercise 4. Verify (12) by direct computation, using (11).

C. Differential equations.

This section summarizes basic facts about first and second order differential equations. The facts about linear homogeneous differential equations presented in item **C.iv.** have extensions to higher order equations—see, Chapter 3 of Greenberg—but it suffices to review the first and second order cases.

C.i. Separable first order equations

A first order differential equation in the form

$$\frac{dy}{dx} = g(x)f(y) \tag{13}$$

is said to be *separable*. To find the general solution, one multiplies both sides of the equation by $dx/f(y)$ to obtain

$$\frac{1}{f(y)}\,dy = g(x)\,dx.$$

This implies that

$$\int \frac{1}{f(y)}\,dy = \int g(x)\,dx.$$

If the integrals can be calculated, the result is an equation defining y implicitly as a function of x. One must remember to include an arbitrary constant of integration when integrating.

Example. $\dfrac{dy}{dx} = x^3y^2$. The separated equation has the form $y^{-2}\,dy = x^3\,dx$. Integrating both sides yields, $-y^{-1} = (1/4)x^4 + C$, or $y = -1/(x^4/4 + C)$.

C.ii. Linear first order ODE (by integrating factors)

To solve

$$y'(x) + p(x)y(x) = h(x), \tag{14}$$

multiply both sides by the integrating factor $\mu(x) = \exp\{\int_{x_0}^{x} p(t)\,dt\}$. Since $(\mu(x)y(x))' = \mu(x)(y'(x) + p(x)y(x)) = \mu(x)h(x)$, we can integrate both sides (and use $\mu(x_0) = 1$) to find

$$\mu(x)y(x) - y(x_0) = \int_{x_0}^{x} \mu(t)h(t)\,dt.$$

Setting $A = y(x_0)$, solving for $y(x)$, and inserting the definition of $\mu(x)$ gives the general solution:

$$y(x) = \exp\{-\int_{x_0}^{x} p(t)\,dt\}\left[A + \int_{x_0}^{x} \exp\{\int_{x_0}^{t} p(z)\,dz\}h(t)\,dt\right]. \tag{15}$$

For example, the general solution to $y' + 2xy = h(x)$ is, taking $x_0 = 0$ in (15) ,

$$y(x) = Ae^{-x^2} + \int_{0}^{x} e^{y^2-x^2}h(t)\,dt.$$

C.iii. Second order, homogeneous, linear differential equations: general case. (See Greenberg, 3.1—3.3.) These are equations of the form

$$y''(x) + p_1(x)y'(x) + p_2(x)y(x) = 0. \qquad (16)$$

The term "homogeneous" refers to the fact that the right-hand side is 0.

We are interested in the set of all solutions to this equation. Here are the principal facts about this set.

(a) If $y_1(x)$ and $y_2(x)$ are both solutions to (16) then so is any linear combination $c_1y_1(x)+c_2y_2(x)$.

(b) Equation (16) has two linearly independent solutions. This means that there exist two non-zero solutions y_1 and y_2 that are linearly independent in the sense y_2 is not a constant multiple of y_1, i.e., $y_2(x) \neq cy_1(x)$ for all x, for any c.

(c) If y_1 and y_2 are two linearly independent solutions of (16), then any other solution can be written in the form $c_1y_1(x)+c_2y_2(x)$; that is, $c_1y_1(x)+c_2y_2(x)$ is an expression for the general solution to (16).

C.iv. Second order, homogeneous, linear differential equations: constant coefficient case. (See Greenberg, 3.4.)

The equation is now

$$y'' + \alpha y' + \beta y = 0 \qquad (17)$$

The general solution can be found explicitly. We look for solutions of the form e^{rx}. By plugging into (17), we find that this will be a solution if and only if $(r^2 + \alpha r + \beta)e^{rx} = 0$, which is true if and only if r satisfies the *characteristic equation*

$$r^2 + \alpha r + \beta = 0. \qquad (18)$$

There are three cases:

(a) The characteristic equation has two, unequal real roots r_1, r_2. Then e^{r_1x} and e^{r_2x} are linearly independent solutions of (17) and the general solution is $c_1e^{r_1x} + c_2e^{r_2x}$.

(b) The characteristic equation has two complex roots $r_1 = a+bi$ and $r_2 = a-bi$. Again, $e^{r_1x} = e^{ax}\cos(bx) + ie^{ax}\sin(bx)$ and $e^{r_2x} = e^{ax}\cos(bx) - ie^{ax}\sin(bx)$ are linearly independent solutions, but they are complex-valued. However, the real and imaginary parts $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$ are linearly independent, real-valued solutions, and the general solution can be written as $c_1e^{ax}\cos(bx) + c_2e^{ax}\sin(bx)$.

(c) The characteristic equation has a single, real root r_1. In this case e^{r_1x} and xe^{r_2x} are linearly independent solutions of (17) and the general solution can be written as $c_1e^{r_1x} + c_2xe^{r_1x}$.

Exercise 5. Find the general solution of each of the following differential equations:

$$(a)\ y` + \frac{2}{x}y = 3x + \ln x\ ; \qquad (b)\ (\sec x)\, y' = \frac{y}{y+1}\ ;$$
$$(c)\ y'' + 3y' + 2y = 0\ ; \qquad (d)\ y'' + 2y' + 3y = 0\ .$$

In each case your general solution should contain only real-valued functions.

D. Eigenvalues and eigenvectors. Given an $n \times n$ matrix A and an n-vector $\mathbf{x}$, recall the matrix-vector product

$$A\mathbf{x} = \begin{pmatrix} a_{11} & \dots & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_1^n a_{1j}x_j \\ \vdots \\ \sum_1^n a_{nj}x_j \end{pmatrix}.$$

A number λ, real or complex, is called an *eigenvalue* of A if there exists a non-zero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda \mathbf{x}. \tag{19}$$

A non-zero vector satisfying this equation is called an *eigenvector* corresponding to λ.

The eigenvector-eigenvalue equation (19) is equivalent to

$$(A - \lambda I)\, \mathbf{x} = \mathbf{0}, \tag{20}$$

where I denotes the $n \times n$ identity matrix (1's on the diagonal, 0's elsewhere), and $\mathbf{0}$ is the n-vector of all 0 entries.

We will need to compute eigenvectors and eigenvalues mostly for 2×2 matrices. The procedure which review is however general. The theory of matrices says that (20) has non-zero solutions if and only if $\det(A - \lambda I) = 0$. So to find the eigenvalue one need only compute $\det(A - \lambda I)$, which is always a polynomial of degree n, and find its roots. Then, once an eigenvalue is found, substitute it in (20) and use Gaussian elimination to find all solutions $\mathbf{x}$.

Example. Find all eigenvalues and their corresponding eigenvectors of

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.$$

The equation for the eigenvalues is

$$0 = \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} = \lambda^2 - 5 = 0.$$

The eigenvalues are $\lambda_1 = \sqrt{5}$ and $\lambda_2 = -\sqrt{5}$. To find the eigenvectors corresponding to $\lambda_1 = \sqrt{5}$ one must find all non-zero solutions to

$$A = \begin{pmatrix} 2-\sqrt{5} & 1 \\ 1 & -2-\sqrt{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or equivalently,

$$\begin{aligned}(2-\sqrt{5})x_1 + x_2 &= 0\\ x_1 + (-2-\sqrt{5})x_2 &= 0\end{aligned}$$

However, the fact that the determinant of the coefficient matrix is 0 implies that the second equation is a multiple of the first and so any solution to the first equation will satisfy the second automatically; this is always the case with 2×2 matrices, but not with higher dimensions. (In this instance, one can see easily that the two equations are equivalent because $(2-\sqrt{5})\left[x_1 + (-2-\sqrt{5})x_2\right] = (2-\sqrt{5})x_1 + x_2$.) Thus we need only find all solutions to the first equation. Setting $x_1 = c$ to an arbitrary constant, we find that all solutions have the form

$$\begin{pmatrix} c \\ -c(2-\sqrt{5}) \end{pmatrix} = c\begin{pmatrix} 1 \\ \sqrt{5}-2 \end{pmatrix}$$

This will be an eigenvector if $c \neq 0$. Hence the set of eigenvectors corresponding to $\lambda_1 = \sqrt{5}$ is the set of non-zero multiples of $(x_1, x_2) = (1, \sqrt{5}-2)$.

Similarly, the eigenvectors corresponding to $\lambda_2 = -\sqrt{5}$ are the non-zero multiples of $(x_1, x_2) = (1, -\sqrt{5}-2)$.

Exercise 6. Find the eigenvalues of the matrix

$$\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix},$$

and find an eigenvector for each eigenvalue.