Mathematics 535 Algebraic Geometry Fall 2008

Problem Set 3A (Last revised 9/30/2008)

Recall that a morphism $\phi : X \to Y$ for projective varieties $X, Y$ is a continuous function such that for each open set $U$ of $Y$ and regular function $f$ on $U$ the composition $f \circ \phi$ is regular on $\phi^{-1}(U)$.

Let $K$ be an algebraically closed field.

1. Verify that if $X \subset \mathbf{P}^m$ is a projective variety and $F_i(x_0, \ldots, x_m), i = 0, \ldots, n$ are homogeneous polynomials with no common zero on $X$ then the function $\phi([x_0, \ldots, x_m]) = [F_0(x_0, \ldots, x_m), \ldots, F_n(x_0, \ldots, x_m)]$ is a well defined function from $X$ to $\mathbf{P}^n$. By considering the restriction of $\phi$ to the open subsets of the affine variety $X \cap \{[x_0, \ldots, x_m] | x_j \neq 0\}$ where $F_i(x_0, \ldots, n_m) \neq 0$ show that $\phi$ is a morphism from $X$ to $\mathbf{P}^n$ (see the middle paragraph on page 21 of Harris).
2. The goal of this problem is to determine the set of morphisms from $X = \mathbf{P}^1$ to $Y = \mathbf{P}^1$. Take homogeneous coordinates $[x_0, x_1]$ on $X$ and $[s, t]$ on $Y$.
   a) Suppose that r(z) is a rational function of the variable z, that is a member of the fraction field $K(z)$ of the polynomial ring $K[z]$ ($K(z)$ is the localization of $K[z]$ with respect to the multiplicative set of nonzero polynomials). If $r(z) = f(z)/g(z)$ is an expression for the rational function where $f(z), g(z)$ have no common root, define the degree d of $r(z)$ to be the maximum of degrees of $f(z), g(z)$. Show that the rational functions $F = x_0^d f(x_1/x_0), G = x_0^d g(x_1/x_0)$ are homogeneous polynomials in $x_0, x_1$ of degree d, which factor as the product of d linear polynomials, and have no common zeros on $P^1$. Let $\phi_r([x_0, x_1]) = [F(x_0, x_1), G(x_0, x_1)]$ be the morphism from $\mathbf{P}^1$ to $\mathbf{P}^1$ defined in problem 1.
   b) Show that the construction of (a) gives a one to one map of the monoid of rational functions under composition to the monoid of morphisms from $\mathbf{P}^1$ to $\mathbf{P}^1$.
   c) Show that every morphism $\phi$ from $\mathbf{P}^1$ to itself arises from the construction in (a). Hint: on some nonempty open set in $X$ show that $\phi([x_0, x_1])$ is a rational function $r$ of $x_1/x_0$ and verify that $\phi = \phi_r$.
   d) Use the preceeding to show that the group of automorphisms $Aut(\mathbf{P}^1)$ is isomorphic to the group of degree 1 rational functions under composition and that this is isomorphic to $PGL(2, K)$.

1.27 Show that the images of the maps $\mu, \nu : \mathbf{P}^1 \to \mathbf{P}^2$ given by $\mu[x_0, x_1] = [x_0^3, x_0 x_1^2, x_1^3]$ and $\nu[x_0, x_1] = [x_0^3, x_0 x_1^2 - x_0^3, x_1^3 - x_0^2 x_1]$ are algebraic varieties.

1.28 Let $\nu : \mathbf{P}^1 \to \mathbf{P}^2$ be given by three homogeneous cubic polynomials. Show that if the polynomials have no common zero, then the image is a hypersurface which is the zero set of a cubic polynomial.

1.29 Let $\nu_{\alpha,\beta} : \mathbf{P}^1 \to \mathbf{P}^3$ be given by $\nu_{\alpha,\beta}([x_0, x_1]) = [x_0^4 - \beta x_0^3 x_1, x_0^3 x_1 - \beta x_0^2 x_1^2, \alpha x_0^2 x_1^2 - x_0 x_1^3, \alpha x_0 x_1^3 - x_1^4]$. Show that the image of this map is a projective variety which is the zero locus of one quadratic and two cubic polynomials.