Problem Set 4B (Last revised 10/9/2008)

3.9 Show that the curves of exercise 1.29 may be realized as projections of a rational normal curve in $\mathbf{P}^4$ from points $p_{\alpha,\beta} \in \mathbf{P}^4$ and use these to show that the projections $\pi_p(X), \pi_{p'}(X)$ of a variety X from different points need not be projectively equivalent.

4.2 Let $X \subset \mathbf{P}^n$ be a projective variety and let $\{V_b\}$ be a family of projective varieties in $\mathbf{P}^n$ with base B. Show that the set

$$\{b \in B : X \cap V_b \neq \emptyset\}$$

is closed in B. In general, if $\{W_b\}$ is another family of projective varieties in $\mathbf{P}^n$ with base B show that the set $\{b : V_b \cap W_b \neq \emptyset\}$ is a closed subvariety of B.

4.3 With the notation of 4.2, let $\{W_c\}$ be another family of projective varieties in $\mathbf{P}^n$ with base C. Show that the set

$$\{(b, c) : V_b \cap W_c \neq \emptyset\}$$

is a closed subvariety of $B \times C$, and show that this implies the statements in exercise 4.2

4.4 Show that for each subvariety $X \subset \mathbf{P}^n$ and any family $\{V_b\}$ of subvarieties in $\mathbf{P}^n$ the subset $\{b \in B : X \subset V_b\}$ is closed in B.

4.10 Consider $\mathbf{P}^N$ as the parameter space of hypersurfaces of degree d in $\mathbf{P}^n$, where $N = \binom{n+d}{d} - 1$. Show that the subset $\Sigma \subset \mathbf{P}^N$ consisting of nonprime polynomials $F(X_0, \ldots, X_N)$ is a projective subvariety of $\mathbf{P}^N$. In case $d = 2, n = 2$ identify this subvariety. We will show later that in general it is the collection of hypersurfaces of degree d containing a hypersurface of strictly smaller degree.

4.11 Let $X \subset \mathbf{P}^n$ be any hypersurface of degree d given by a homogeneous polynomial $F(Z_0, Z_1, \ldots, Z_n)$. Show that the subset of hyperplanes $H \subset \mathbf{P}^n$ such that the restriction of F to H factors is a subvariety of $\mathbf{P}^n$ (hence the set of hyperplanes H such that $H \cap X$ contains a hypersurface of degree $< d$ form a subvariety).