Problem Set 5A (Last revised 10/28/2008)

2.18 Let $C_{\alpha,\beta}$ be the curve of problem 1.29 (a so-called rational quartic curve). Show that $C_{\alpha,\beta}$ lies on the Segre surface S which is the zero set of $Z_0Z_3 - Z_1Z_2$ in $\mathbf{P}^3$. Show that S is the unique quadric surface containing $C_{\alpha,\beta}$, and that $C_{\alpha,\beta}$ is the zero set of a bihomogeneous polynomial of bidegree $(1,3)$ on $S \simeq \mathbf{P}^1 \times \mathbf{P}^1$. Use this to give an alternate solution to problem 1.29.

2.19 Use 2.18 to show that there is a continuous family of curves $C_{\alpha,\beta}$ not projectively equivalent to one another.

4.16 Let $U \subset \mathbf{P}^n \times \mathbf{P}^n$ be the complement of the diagonal. Let $\Omega = \{(p,q,r) : r \in \overline{pq}\} \subset U \times \mathbf{P}^n$. Show that Ω is a subvariety of $U \times \mathbf{P}^n$. More generally show that for any k the subset $U \subset (\mathbf{P}^n)^k$ formed by the points $(p_1,\dots,p_k)$ such that $p_1,\dots,p_k$ are linearly independent is open and the set

$$\Omega = \{(p_1,\dots,p_n;r) : r \in \overline{p_1 \dots p_n}\} \subset U \times \mathbf{P}^n$$

is a subvariety. Find the closure of Ω in $(\mathbf{P}^n)^k \times \mathbf{P}^n$.

5.2 Show that he following are equivalent for homogeneous ideals $I, J \subset K[Z_0,\dots,Z_n]$

$$\begin{cases} (i) & I \text{ and } J \text{ have the same saturation} \\ (ii) & I_m = J_m \text{ for all } m \text{ sufficiently large} \\ (iii) & I, J \text{ generate the same ideal in each localization } K[Z_0,\dots,Z_n,Z_i^{-1}] \end{cases}$$

5.13 Show that for any $d, n \leq (d+1)(d+2)/2$ a general set of n points in $\mathbf{P}^2$ imposes independent conditions on curves of degree d. This means that the space of polynomials vanishing at the points has codimension n in the space of all homogeneous polynomials of degree d. In case $n \leq 2d+1$ what open set in $(\mathbf{P}^2)^n$ is implicitly referred to here.

5.14 Let $C \subset \mathbf{P}^n$ be a curve of degree d, that is the zero set of a homogeneous polynomial $F(Z_0, Z_1, Z_2)$ of degree d, which has no repeated factors in its unique factorization into irreducibles. Show that the general line in $\mathbf{P}^2$ will intersect C in d points.

6.2 Suppose that the characteristic of K is not 2. Show that a vector $\omega \in \bigwedge^2 V$ is decomposable if and only if $\omega \wedge \omega = 0$ and hence the Grassmannian G(2,V) of 2 planes in V is cut out by quadrics in $\mathbf{P}(\bigwedge^2 V)$.