Add to Exercises 1.7.3:

5. On $G = \mathbb{C}^\times$ define the conjugation $\tau(z) = \bar{z}^{-1}$. Let $V \subset \mathcal{O}[G]$ be the subspace with basis $f_1(z) = z$ and $f_2(z) = z^{-1}$. Define $Cf(z) = \overline{f(\tau(z))}$ and $\rho(g)f(z) = f(zg)$ for $f \in V$ and $g \in G$, as in the proof of Theorem 1.7.3.

(a) Find a basis $\{v_1, v_2\}$ for the real subspace $V_+ = \{f \in V \,:\, Cf = f\}$ so that the matrix of $\rho(z)$ relative to this basis is

$$\begin{bmatrix} (z+z^{-1})/2 & (z-z^{-1})/2i \\ -(z-z^{-1})/2i & (z+z^{-1})/2 \end{bmatrix}.$$

(b) Let $K = \{z \in G \,:\, \tau(z) = z\}$. Use **(a)** to show that $G \cong \mathrm{SO}(2,\mathbb{C})$ as an algebraic group and $K \cong \mathrm{SO}(2)$ as a Lie group.