Review problems for Final Exam in 642:550 in Fall 2003

These problems are taken from past final exams and problems in the text. The final exam will be similar in form to this collection of problems. The final covers problems from the entire semester.

1. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix}$$

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(a) Show that the system $Ax = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ is inconsistent.

(b) Find the singular value decompositon $A = Q_1 \Sigma Q_2^T$.

(c) Find the pseudoinverse of A.

(d) Find the vector z of least norm such that Az is the closest vector to $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ by using the pseudoinverse of A.

(e) Write A as a sum of rank 1 matrices $A = \sum \alpha_i u_i v_i^T$ where the α_i are singular values of A.

2. Let

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}$$

a) Find the characteristic polynomial and eigenvalues of A.

b) For each eigenvalue λ of A, find the corresponding eigenspace.

(c) Use the information from (a) and (b) to write down the Jordan form J of A (you do not have to find the matrix M so that $J = M^{-1}AM$.)

(d) Optional extra credit problem: Find a basis of $\mathbf{R^3}$ consisting of Jordan strings for the matrix A and find a matrix M so that $M^{-1}AM$ is in Jordan canonical form.

3.) Give a counterexample or explain why the statement is true:

a) A matrix has a pseudoinverse if and only if it is not invertible.

b) The matrix $\begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}$ is similar to a diagonal matrix.

c) (6) If A is a 5×5 matrix with integer entries and $\det A = 1$, then every entry in A^{-1} is an integer.

d) A square matrix of the form $Q^{-1}DQ$ for D diagonal is symmetric if and only if Q is orthogonal.

e) The kth powers of the matrix $M = \begin{pmatrix} .3 & .1 & .2 \\ .4 & .5 & .5 \\ .3 & .4 & .3 \end{pmatrix}$ approach a limit as $k \to \infty$

f) An orthogonal set of n nonzero vectors in $\mathbf{R^n}$ form a basis of $\mathbf{R^n}$.

4. Let

$$B = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Find the eigenvalues of A. For each eigenvalue find a corresponding eigenvector.
(b) Find a diagonal matrix D and an invertible matrix S so that $S^{-1}AS = D$. (You do not have to compute S^{-1} or the product)
(c) Find e^{Bt} for t a scalar by using (b)
(d) Find all solutions to the system of differential equations

$$\frac{dy}{dt} = By, y(0) = \begin{pmatrix} 6 \\ 0 \\ 9 \end{pmatrix}$$

5. Let s be a real number and define

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & s \end{pmatrix}$$

(a) (8) For what values of s is A positive definite? Justify your answer.
(b) (8) Find a diagonal matrix D congruent to A.
(c) (9) Use the result of (b) and the law of inertia to determine the number of positive, zero, and negative eigenvalues of A (your answers will depend on s).

6. Let $v = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Define a linear transformation H from $\mathbf{R^3}$ to $\mathbf{R^3}$ by $Hx = x - (v^T x)v$.
(a) Calculate He_1, He_2, He_3 where e_j is the standard basis vector for $\mathbf{R^3}$.
(b) Find the matrix for H relative to the standard basis.
(c) Calculate Hv and Hx when $x \perp v$. Use this information to find the eigenvalues and eigenspaces of H.

7. Let $R = \begin{pmatrix} a & d & f \\ 0 & b & e \\ 0 & 0 & c \end{pmatrix}$ be an upper triangular 3×3 matrix. Assume that R is invertible. Set $A = R^T R$.
(a) Prove that A is positive definite.
(b) Find a formula for $\det A$ in terms of the entries of R.
(c) Prove that $\det A \leq a_{11}a_{22}a_{33}$ where the a_{ij} are entries in A.
(d) Is the result in (c) true for every positive definite 3×3 matrix A? Justify your answer.

8. Let

$$A = \begin{pmatrix} t & 1 & 0 \\ 1 & t & 1 \\ 0 & 1 & t \end{pmatrix}$$

where t is a complex number. Assume that t is taken so that $\det A \neq 0$.

(a) Calculate $\det A$, the cofactor matrix A_{cof} and A^{-1}.

(b) Use Cramer's rule to solve the equation

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Express your answers in terms of rational functions of t.