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LAB 2: Complete Solution to $A\mathbf{x} = \mathbf{b}$

In this lab you will use MATLAB to study the complete solution of the linear equation $A\mathbf{x} = \mathbf{b}$, where A is a given $m \times n$ matrix, $\mathbf{b}$ is a given $m \times 1$ column vector and $\mathbf{x}$ is an unknown $n \times 1$ column vector. This will involve the *four fundamental subspaces* associated with the matrix A.

MATLAB Preliminaries

Tcodes: A special set of MATLAB *m-files* called *Tcodes* has been written to accompany Strang's textbook. To obtain any of these files, use a web browser (such as Netscape) and go to the Math Department Home page *http://www.math.rutgers.edu*. Click on *course materials*, then on *Math 550A Linear Algebra and Applications*, and then on *MATLAB Teaching Codes*. You will see a directory of the m-files. Click on the particular m-file that you need. Then in the menu bar click on *Files* and *Save As*. Fill in the directory information that is requested.

For this lab you will need the Teaching Codes

```
nulbasis.m, elim.m, partic.m
```

Before beginning work on the Lab questions you should copy these codes to your diskette (or hard drive if available), as described above.

Random Seed: When you start your MATLAB session, initialize the random number generator by typing

```
rand('seed', abcd)
```

where *abcd* are the last four digits of your Student ID number. This will ensure that you generate your own particular random vectors and matrices.

BE SURE TO INCLUDE THIS LINE IN YOUR LAB WRITE-UP

Question 1. Reduced Row Echelon Form (RREF)

(a) Random Square Matrices: The Teaching Code `elim` calculates the reduced row echelon form matrix R for A and the invertible elimination matrix E so that $EA = R$. Type

```
A = rmat(5, 5), [E, R] = elim(A)
```

at the prompt. Then calculate $E * A - R$ and check that the result is approximately zero. (Throughout these labs whenever your are asked to verify that some MATLAB calculation returns zero as an answer, this is what is meant. Most MATLAB algorithms that you use in these assignments will have round off errors ϵ in the range of 10^{-14}. For the `elim` algorithm, however, the matrix entries of $E * A - R$ will be on the order of magnitude $\epsilon = 10^{-4}$ due to the large round-of errors in the `rref` algorithm, as you saw in Lab 1.) The *rank* of A is the number of pivots in R. Determine this number, and then check by the MATLAB command `rank(A)`. Use the up-arrow key ↑ to repeat these calculations.

Based on this evidence, what do you expect to get as the reduced row echelon form R of a *completely random* 5×5 matrix A? What do you expect for `rank(A)`?

(b) Random Thin Matrices: Now we study matrices A with more rows than columns. Give the command

```
A = rmat(6, 3), [E, R] = elim(A)
```

(you can do this by using ↑ and editing the command from part (a)). Calculate `inv(E)`.

Repeat these calculations using ↑. How many pivots are there in each case? Which columns are pivot columns? How is E related to A?

Based on this evidence, what do you expect to get as the reduced row echelon form R of a *completely random* 6×3 matrix A? What do you expect for `rank(A)`?

(c) Random Fat Matrices: Now consider matrices A with more columns than rows. Give the command

```
A = rmat(3, 6), [E, R] = elim(A)
```

Repeat this calculation using the up-arrow key. Which entries in the matrix R are the same both times? Which are different? Which columns of R contain the pivots?

Now calculate

```
B = A(:,1:3), E*B
```

Describe in words how the square matrix B is formed from rectangular matrix A. What is the relation between B and the elimination matrix E? Calculate `rank(A)`.

Based on this evidence, what would you expect to get as the reduced row echelon form R of a *completely random* 3×6 matrix A? Which columns of R would you know before any computation? Which columns of R are unpredictable until after the computation? How many columns of R will have pivots? What do you expect for `rank(A)`?

(d) Partly Random Fat Matrices: The built-in MATLAB command `rref(A)` also calculates the reduced row echelon form of A (but does not return the elimination matrix E). Generate a matrix A and its reduced row echelon form R by

```
B = rmat(3, 2); C = rmat(3,2); A = [B, 3*B, C], R = rref(A)
```

Use the up-arrow key to repeat this calculation. Which entries in the matrix R are the same both times? Which are different? What are the positions of the pivots in R? How many columns of R have pivots?

Does R fit the pattern that you predicted in part (c) for the RREF of a *completely random* 3×6 matrix? Explain how columns 3 and 4 of A are determined by columns 1 and 2 of A. How does this explain the entries in columns 3 and 4 of R?

Question 2. Nullspace of A

(a) Special Solutions to $A\mathbf{x} = 0$ (A fat and completely random): The MATLAB Teaching Code `nulbasis.m` calculates the *special solutions* to the equation $A\mathbf{x} = 0$. Try it first with

```
A = rmat(3, 6), R = rref(A), N = nulbasis(A)
```

The columns of N are the *special solutions* to $Ax = 0$. Check that $A * N = 0$ by MATLAB.

Define

```
s1 = N(:,1),   s2 = N(:,2),   s3 = N(:,3)
```

Which component of $\mathbf{s}_1$ *must be* 1? Which components of $\mathbf{s}_1$ *must be* 0? Answer the same questions for $\mathbf{s}_2$ and $\mathbf{s}_3$

Which columns of R correspond to the free variables? From Question 1 you know that R will have the block matrix form

$$R = [I \quad F], \quad \text{where } I = 3 \times 3 \text{ identity matrix}, \quad F = 3 \times 2 \text{ matrix}$$

(since A is a *general* 3×6 matrix). Use MATLAB to obtain N from F. (*Hint:* Look at part **(c)** of Question 1).

(b) General Solution to $A\mathbf{x} = 0$ (A fat and partly random): Generate a *partly random* 3×6 matrix A by

```
B = rmat(3, 2); C = rmat(3,2);
A = [B, 3*B, C], R = rref(A), N = nulbasis(A)
```

Which columns of R contain the pivots? Which columns of R correspond to the free variables?

Define the *special solutions* to $A\mathbf{x} = 0$ as in part **(a)**:

```
s1 = N(:,1),   s2 = N(:,2),   s3 = N(:,3)
```

Now generate a random linear combination of the vectors $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$ by

```
x = rand(1)*s1 + rand(1)*s2 + rand(1)*s3
```

(Each occurrence of `rand(1)` generates a different random coefficient). Check by MATLAB that $A\mathbf{x} = 0$. Explain (without MATLAB) why $\mathbf{x}$ also satisfies $R\mathbf{x} = 0$. Then verify this by MATLAB.

(c) General Solution to $A\mathbf{x} = 0$ (A completely random and thin): When A is thin, then the equation $A\mathbf{x} = 0$ is *overdetermined* (more equations than unknowns). In general there will not be any solution other than the *trivial solution* $\mathbf{x} = 0$. Try

```
A = rmat(6, 3), R = rref(A), N = nulbasis(A)
```

Are there any columns of R corresponding to free variables? What vectors are in the null space of A?

Question 3. Solving $A\mathbf{x} = \mathbf{b}$

(a) Particular Solution (A fat and completely random): Generate a random 3×5 matrix A and its reduced row echelon form R by

```
A = rmat(3,5),   R=rref(A)
```

Now generate a random 3×1 vector $\mathbf{b}$ and use the Teaching Code `partic.m` to find a particular solution to $A\mathbf{x} = \mathbf{b}$ by

```
b = rmat(3,1),   x = partic(A, b)
```

Check that $A\mathbf{x} = \mathbf{b}$. Repeat for another random vector $\mathbf{b}$, using the same matrix A. What entries in x are zero both times? Which columns of R correspond to free variables? Calculate `rank(A)` and `rank([A, b]` (the augmented matrix). Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for *every* possible $\mathbf{b}$ in this case? Why?

(b) Particular Solution (A thin and completely random): Generate a random 5×3 matrix `A = rmat(5,3)`. The following MATLAB command will generate a random 5×1 vector $\mathbf{b}$ and try to find a particular solution to $A\mathbf{x} = \mathbf{b}$:

```
b = rmat(5,1),   x = partic(A, b)
```

What did MATLAB return for $\mathbf{x}$? Calculate `rank(A)` and `rank([A, b])`. Do you get the same relation between the ranks as you got in part (a)? Use this information to explain why there is no solution to $A\mathbf{x} = \mathbf{b}$ for a completely random choice of $\mathbf{b}$.

Now use the (partly) random vector

```
b = rand(1)*A(:,1) + rand(1)*A(:,2) + rand(1)*A(:,3)
```

and calculate `x = partic(A, b)` . Explain why the special form of $\mathbf{b}$ guarantees that there always is a solution $\mathbf{x}$, no matter what the random coefficients might be. Since there is a solution to $A\mathbf{x} = \mathbf{b}$ for this $\mathbf{b}$, what is the rank of the augmented matrix `[A, b]`? Answer first without calculation, and then check using MATLAB.

(c) General Solution (A fat and completely random): Execute the commands

```
A = rmat(3,5), b = rmat(3,1), N = nulbasis(A), xp = partic(A,b)
```

Set `s1 = N(:,1), s2 = N(:,2)` and form a random *general solution*

```
x = xp + rand(1)*s1 + rand(1)*s2
```

to $A\mathbf{x} = \mathbf{b}$. Check by MATLAB that $A\mathbf{x} - \mathbf{b} = 0$. Use ↑ to get a different random general solution $\mathbf{x}$ (for the same A and $\mathbf{b}$). Check again by MATLAB that this vector satisfies $A\mathbf{x} - \mathbf{b} = 0$.

Finally, solve the equation $A\mathbf{x} = \mathbf{b}$ with the extra condition that $\mathbf{x}$ should be of the form

$$\mathbf{x} = [x_1,\, x_2,\, x_3,\, -9,\, 8]^{\mathrm{T}}$$

For this, you must choose particular scalars c_1 and c_2 so that

$$\mathbf{x} = \mathbf{x}_p + c_1\mathbf{s}_1 + c_2\mathbf{s}_2$$

(*Hint:* look at the free variables in $\mathbf{x}$, $\mathbf{x}_p$, $\mathbf{s}_1$, and $\mathbf{s}_2$). Then check your answer by calculating $A\mathbf{x} - \mathbf{b}$ with MATLAB.

Question 4. The Four Fundamental Subspaces

(a) Generate a random 3×5 matrix `A = rmat(3,5)`. The MATLAB function `orth` will produce an orthonormal basis for the column space $\mathcal{R}(A)$. Let `R = orth(A)`. Verify that the columns of R are an orthonormal set of vectors by calculating `R'*R`. What is the rank of A?

(b) What is the dimension of the null space $\mathcal{N}(A)$? The MATLAB function `null` will produce an orthonormal basis for $\mathcal{N}(A)$. Let `N = null(A)`. Verify that the columns of N are an orthonormal set of vectors by calculating `N'*N`. Also verify that $A * N = 0$, so each column of N is in the null space of A.

(c) Let `RT = orth(A')` and `NT = null(A')` be the column space and the null space of A^T. Use the theory of the *Four Fundamental Subspaces* associated with A to determine which of the following matrices must be zero:

```
N'*R,   N'*RT,   NT'*R,   NT'*RT .
```

Now check your answer by MATLAB.