Review problems for midterm in 642:550 in Fall 2003

These problems are taken from past midterm exams and problems in the text. The midterm will be similar in form to this collection of problems.

1. (The four fundamental subspaces associated to a matrix, their dimension and a basis for each) Let V be the subspace spanned by

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

   (a) (4) Find a square matrix A that has V as its row space. and a square matrix B that has V as its null space.
   (b) (4) Find the dimensions of the row space of A, the null space of A, the column space of A and the left null space of A.
   (c) (10) Find a basis for the column space of A and a basis for the null space of A, and explain how you obtained them.
   (d) (6) Find the echelon form for A^T and use it to find a basis for the left null space of A.
   (e) (4) Suppose that $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ is in the column space of A. What homogeneous linear equations must y_1, y_2, y_3 satisfy?

2. (16) Let

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

   Solve the system

$$LU \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 11 \\ 4 \end{pmatrix}$$

   without multiplying the matrices L, U to find the coefficient matrix.

3.) (12) Give a counterexample or explain why the statement is true:
   a) If A is the coefficient matrix of a system of 13 equations which has free variables, then the equation $Ax = y$ has a solution for any $y \in \mathbf{R}^{\mathbf{13}}$.
   b) If A is a wider than long matrix, then the equation $Ax = 0$ has a nonzero solution.
   c) Any 9 vectors in a vector space of dimension 8 are linearly dependent.
   d) If A is a 5 X 3 matrix and the columns of A are linearly independent, then $\text{rank}(BA) = \text{rank}(B)$ for every 5 x5 matrix B.
   e) If S and T are subspaces of $\mathbf{R}^{\mathbf{10}}$ with $\dim S = 7, \dim T = 5$ then $\dim(S \cap T) \geq 2$.
   f) If S and T are any subspaces of $\mathbf{R}^{\mathbf{10}}$ with $\dim S = 7, \dim T = 3$, then $S + T = \mathbf{R}^{\mathbf{10}}$.

4. (6)
   a) Find a basis for the subset of all vectors in $\mathbf{R^3}$ whose first two components are equal. Give details of why your set is a basis.
   b) Is the following set of vectors a linearly independent or linearly dependent set? Give reasons supporting your conclusion.

$$\left\{ \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}$$

5. Suppose that $A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix}$.
   (a) (6) Compute the projection of $b = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$ onto the column space of A.
   (b) (4) Compute the decomposition of b as $p + q$ where p is in the column space of A and q is perpendicular to that space. In which of the 4 subspaces associated to A does the vector q lie? Explain.
   (c) (3) Find the projection matrix onto the column space of A.
   (d) (3) Use the results of (a) and (b) to find the least-squares approximate solution $\bar{x} = \begin{pmatrix} u \\ v \end{pmatrix}$ to the inconsistent system of equations

$$u + v = 1$$

$$2u - v = 2$$

$$-2u + 4v = 7$$

6. Let

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}$$

(a) (2) Factor $A = QR$ with Q an orthogonal matrix and R upper triangular.
(b) (6) Calculate R^{-1} by the Gauss-Jordan method, and give details.
(c) (4) Set $b = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Solve the equation $Ax = b$ by using the factorization and the results of (a) and (b). Give details.

7. (10) If $A = QR$ is the decomposition resulting from the Gram-Schmidt process applied to the columns of A, find a simple formula for the projection matrix P onto the column space of A.