Mathematics 642:550 **Linear Algebra** **Fall 2003**

SOLUTIONS TO MIDTERM EXAMINATION

1.

(a) Reversing the row operations shows that $A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ -2 & -4 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

(b) From the row echelon form we see that the rank is 2 and thus the nullity is 4-2=2. From this we see all other dimensions are 2, except the nullity of the transpose, which is $3 - \text{rank}(A^T) = 1$.

(c) The pivot columns of A form a basis for the column space, which are the first and last. From the row echelon form the second and third columns are respectively 2 times and -1 times the first.

(d) There are two free variables, from which we obtain a basis $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

2.)

a) The row echelon form of A^T is $\begin{pmatrix} 1 & -2 & 0 \\ 2 & -4 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ from which it follows that the null space of A^T is spanned by $\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

b) The left null space of A and the column space of A are orthogonal to one another, as one sees by computing $\begin{pmatrix} -2 & -1 & 1 \end{pmatrix} A$.

c) Any such b is in the column space of A, so is orthogonal to the null space of A^T. Thus $[-2, -1, 1]b = -2y_1 - y_2 + y_3 = 0$.

3.

(a) F, the matrices rref(A) and A of problem 1 are counterexamples.

(b) T, if $Bx = 0$ then $ABx = x = 0$ so nullity of B is 0, hence rank B is 3. Since $ABx = x$ the column space of A is all of $\mathbf{R^3}$, so rank A is 3.

(c) T, $6 \geq \dim(V + W) = \dim(V) + \dim(W-) \dim(V \cap W) = 7 - \dim(V \cap W)$ so the intersection has dimension at least 1.

(d) T,$A^2 + A + I = 0 = A(A + I) + I = *A + I)A$ so $A(-A - I) = I = (-A - I)A$ and A is invertible.

e) T, the columns of a permutation matrix have only one nonzero entry, a 1, so they are normalized. Distinct columns have the 1's in different places, so the inner product of distinct columns is 0. Thus a permutation matrix has columns forming an orthonormal basis.

f) F, the column space of $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ contains only vectors with last entry 0, while A has rank 3.

4. (a) $A = LU = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ rref(A). By factoring out the elements on the diagonal of U we obtain D is diagonal with $1, 1, 4$ on the diagonal.

(b) $C = D^{1/2}L^T = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

(c) $C^{-1} = \begin{pmatrix} 1 & 1 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{pmatrix}$ by the Gauss-Jordan method.

5.

(a) nullity of A is $2-$rank$(A) = 0$. If $A^TAx = 0$, then $||Ax||^2 = X^TA^TAX = 0$ so that x is in the null space of A.

b) $Ax - y$ orthogonal to columns of A if and only if $A^T(Ax - y) = 0$, that is $A^TAx = A^Ty$. By (a), $x = (A^TA)^{-1}A^Ty$. The projection of y is $Ax = A(A^TA)^{-1}A^Ty$.

(c) Row reduce to obtain an inconsistent system

(d) The normal equation is $A^TA\begin{pmatrix} u \\ v \end{pmatrix} = A^T\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ which has solution $u = 1/2.v = 1/6$. 6.

a) Any set of 5 vectors in a 4 dimensional space are dependent. An explicit relation can be found by solving the system $c1(x^3 + x + 1) = c_2(x^3 + x^2 + 1) + c_3(x^3 - x) + c_4(x^2 - x) + c_5(x + 1) = 0$ by elimination.

b) The remark above reduces this to finding the pivot columns in the coefficient matrix of the system, which gives a basis of 4 polynomials spanning the same space, so no smaller set can span.

7.

(a) The q_i are columns of $Q = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

(b) Since Q is orthogonal $R = Q^TA = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

(c) Since the column space of the matrix is the same as the column space of Q which is all of $\mathbf{R^3}$ the projection is the identity, which is exactly what the formula $A(A^TA)^{-1}A^T$ gives.