I Always true, always false, or sometimes true and sometimes false.

For each of the subparts of this question, the statement given is either always true, always false, or true in some cases and false in others. State which of these is the case by circling the appropriate entry. If the statement is always true or always false. justify your answer by a giving a short proof which may reduce to quoting theorems we have proved. If the statement is sometimes true and sometimes false, give an example where it is true and a second example where it is false, and then state a very closely related result (with only one change from the statement in the exam) which is always true, and give a brief explanation of why it is always true. Almost all of the credit is given for the proofs or examples and closely related theorems

[10] I.1 If $\mathbf{A}$ is a 3×4 real matrix, then the equation $\mathbf{A}\,\mathbf{x} = \mathbf{y}$ always has a nonzero solution $\mathbf{x} \in \mathbb{R}^4$ for any given $\mathbf{y} \in \mathbb{R}^3$.

(a) Always true Always false $\boxed{\text{Sometimes true, sometimes false}}$

(b) Proof or examples and closely related theorem:

If, for example, $\mathbf{A}$ is the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$, or if $\mathbf{A}$ is the zero matrix, or $\cdots$,

you cannot solve the system $\mathbf{A}\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

If, for example, $\mathbf{A}$ is the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ or any rank 3 matrix, one can always solve the equation $\mathbf{A}\mathbf{x} = \mathbf{y}$. For the $\mathbf{A}$ given here, you set $\mathbf{x} = \begin{bmatrix} \mathbf{y} \\ \cdots \\ 0 \end{bmatrix}$.

Possible theorems:

If $\mathbf{A}$ is a 3×4 real matrix *of rank 3*, then the equation $\mathbf{A}\,\mathbf{x} = \mathbf{y}$ always has a nonzero solution $\mathbf{x} \in \mathbb{R}^4$ for any given $\mathbf{y} \in \mathbb{R}^3$.

If $\mathbf{A}$ is a 3×4 real matrix, then the homogeneous equation $\mathbf{A}\,\mathbf{x} = \mathbf{0}$ always has a nonzero solution $\mathbf{x} \in \mathbb{R}^4$.

[10] I.2 Let $F_4\,(i)$ denote the 4×4 Fourier transform matrix. Then $F_4\,(i)$ is a unitary matrix.

(a) Always true $\boxed{\text{Always false}}$ Sometimes true, sometimes false

(b) Proof or examples and closely related theorem: $F_4(i)^H F_4(i) = 4\,\mathbf{I}_5$. $\frac{1}{2}F_4(i)$ is indeed a unitary matrix (but that was not actually asked for here).

[10] I.3 Let $\mathbf{A}$ be an $n \times n$ hermitian matrix over the field of complex numbers, that is, $\mathbf{A} = \mathbf{A}^H$. Then the null space $\mathcal{N}(\mathbf{A})$ of $\mathbf{A}$ is equal to the null space $\mathcal{N}(\mathbf{A}^2)$ of $\mathbf{A}^2$, and the column spaces $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{A}^2)$ are also equal.

(a) $\boxed{\text{Always true}}$ Always false Sometimes true, sometimes false

(b) Proof or examples and closely related theorem:
Proof: Let $\mathbf{Ax} = \mathbf{0}$. Then $\mathbf{A}^H\mathbf{Ax} = \mathbf{A}^2\mathbf{x} = \mathbf{0}$, so $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{N}(\mathbf{A}^2)$. Now let $\mathbf{A}^H\mathbf{Ax} = \mathbf{0}$. Then $\|\mathbf{Ax}\|^2 = \mathbf{x}^H\mathbf{A}^H\mathbf{Ax} = \mathbf{0}$ so $\mathbf{Ax} = \mathbf{0}$ and $\mathcal{N}(\mathbf{A}^2) \subseteq \mathcal{N}(\mathbf{A})$. Hence they are equal.
There are at least two ways to get the statement about the column spaces. One is that by symmetry the left null space of $\mathbf{A}$ is equal to the left null space of $\mathbf{A}^2$ and this is the orthogonal complement of the column spaces of $\mathbf{A} = \mathbf{A}^H$ and $\mathbf{A}^2$. The second is that $\mathcal{R}(\mathbf{A}^2)$ is always a subset of $\mathcal{R}(\mathbf{A})$ and rank plus nullity equals the number of columns of $\mathbf{A}$. Hence the dimension of the subspace $\mathcal{R}(\mathbf{A}^2) \subseteq \mathcal{R}(\mathbf{A})$ is the same as the dimension of the bigger space, so the two spaces must be equal by definition of dimension.

[10] I.4 Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be nonzero vectors in a vector space V over the complex numbers with the standard inner product $\langle \mathbf{x},\ \mathbf{y}\rangle = \mathbf{x}^H\mathbf{y}$. Then for any vector $\mathbf{w} \in V$, the orthogonal projection $\mathbf{P}(\mathbf{w})$ of $\mathbf{w}$ to the subspace of V spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is given by

$$\mathbf{P}(\mathbf{w}) = \frac{\langle \mathbf{v}_1\,,\ \mathbf{w}\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{v}_2\,,\ \mathbf{w}\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

(a) Always true Always false $\boxed{\text{Sometimes true, sometimes false}}$

(b) Proof or examples and closely related theorem:
If $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ clearly $\mathbf{P}(\mathbf{w}) = \mathbf{w}$ whereas $\frac{\langle \mathbf{v}_1\,,\ \mathbf{w}\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{v}_2\,,\ \mathbf{w}\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2$. So here the given equation does not hold.
If $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ the result is clearly true.
The closely related theorem:
Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be nonzero *orthogonal* vectors in a vector space V over the complex numbers with the standard inner product $\langle \mathbf{x},\ \mathbf{y}\rangle = \mathbf{x}^H\mathbf{y}$. Then for any vector $\mathbf{w} \in V$, the orthogonal projection $\mathbf{P}(\mathbf{w})$ of $\mathbf{w}$ to the subspace of V spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is given by

$$\mathbf{P}(\mathbf{w}) = \frac{\langle \mathbf{v}_1\,,\ \mathbf{w}\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{v}_2\,,\ \mathbf{w}\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

II Computation

[20] II.1 Let

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 2 & -2 & 0 \\ 1 & 2 & -2 & 1 \end{bmatrix}$$

For each of the four fundamental subspaces of **A**, find a basis. (Do your major calculations here.)

We put **A** in row echelon form by first permuting rows 1 and 2 and then doing two clearing of columns below the diagonal.

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 2 & -2 & 0 \\ 1 & 2 & -2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & -2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ We can then read off bases of three of the four fundamental subspaces. Notice that these bases have the correct number of elements in them. In this example, one can actually read the answers off without writing out the above computations.

(a) For the subspace *row space* of **A** a basis is: $\left\{ \begin{bmatrix} 1 & 2 & -2 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T \right\}$ where, in this particular case, the first two rows of **A** and the two nonzero rows of the (reduced) row echelon form for **A** are the same. Note that there are two vectors here, as the rank of **A** is 2.

(b) For the subspace *column space* of **A** a basis is: $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, the two columns of **A** that turn out to be pivot columns. You must take these as columns of **A** and not of the row reduced form.

(c) For the subspace *null space* of **A** a basis is: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ since there are two free variables, and you can in turn set one equal to 1 and the other to 0.

(d) For the subspace left null space of **A** a basis is: $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ since one can easily see that this vector is perpendicular to a basis of the column space of **A** and since the nullity of $\mathbf{A}^T$ is $3 - 2 = 1$, this set has nullity of $\mathbf{A}^T$ vectors in it.

[20] II.2 You are given the matrices

$$\mathbf{U} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

Without doing any matrix multiplications (in particular without multiplying $\mathbf{LU}$ to find $\mathbf{B} = \mathbf{LU}$), and without finding the inverse of the matrix $\mathbf{U}$, find the inverse of the matrix $\mathbf{B} = \mathbf{LU}$. Then check your work by doing appropriate matrix multiplications.

You are permitted to compute the inverse of the matrix $\mathbf{L}$, and I believe that you must do so in some form. You may use gaussian elimination, but, as in the Crout algorithm, if you do the arithmetic in a different order, known as forward substitution, you can replace $\mathbf{L}$ by its inverse using no additional space, so I will do that. If you do not like this, use gaussian elimination to compute the inverse. The method of forward substitution, which is standard for programs actually in use for inverting matrices, can then be used to solve for $\mathbf{U}^{-1}\mathbf{L}^{-1}$, but here I will use more traditional gaussian elimination with its need for more space.

To find $\mathbf{L}^{-1}$, you want to solve the systems of equations

$$\mathbf{L}\,\mathbf{L}^{-1} = \mathbf{I}_3$$

Thus you will be solving

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying out the right hand side, we see that $1 \cdot x_1 = 1$, so we will let that 1 in $\mathbf{L}$ stay there. The second equation for column 1 of $\mathbf{L}^{-1}$ says $2x_1 + x_2 = 0$ so $x_2 = -2$ by substituting $x_1 = 1$. We will never need the entry $\mathbf{L}_{2,1}$ again, so we will replace it by $x_2 = -2$. The last equation for column 1 of $\mathbf{L}^{-1}$ is $-2x_1 + x_2 + x_3 = 0$, and we substitute the values for x_1 and x_2 that are now stored in the first two entries of column 1 of the matrix that used to be $\mathbf{L}$ to get $x_3 = -(-2-2) = 4$. The first column of $\mathbf{L}^{-1}$ is now $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ and it has replaced the first column of $\mathbf{L}$. Using the fact that $\mathbf{L}$ is lower triangular, we can do the second and third columns similarly to get

$$\mathbf{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}$$

All operations here are below the diagonal, so the diagonal and above the diagonal can be (and in practice are) used to hold $\mathbf{U}$. This actually means one can compute $(\mathbf{LU})^{-1}$ in the place that originally used to store both $\mathbf{L}$ and $\mathbf{U}$.

Now to find $\mathbf{U}^{-1}\mathbf{L}^{-1}$ we use (the backward pass of) gaussian elimination to solve the equations $\mathbf{U}\ (\mathbf{U}^{-1}\mathbf{L}^{-1}) = \mathbf{L}^{-1}$.

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -1 & 4 & -1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 2 & 4 & 0 & -7 & 2 & -2 \\ 0 & 1 & 0 & 6 & -1 & 2 \\ 0 & 0 & 1 & -4 & 1 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -31 & 6 & -10 \\ 0 & 1 & 0 & 6 & -1 & 2 \\ 0 & 0 & 1 & -4 & 1 & -1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{31}{2} & 3 & -5 \\ 0 & 1 & 0 & 6 & -1 & 2 \\ 0 & 0 & 1 & -4 & 1 & -1 \end{array}\right] \text{ so}$$

$$\mathbf{B}^{-1} = \begin{bmatrix} -\frac{31}{2} & 3 & -5 \\ 6 & -1 & 2 \\ -4 & 1 & -1 \end{bmatrix}$$

To check, multiply

$$\left(\begin{bmatrix} -\frac{31}{2} & 3 & -5 \\ 6 & -1 & 2 \\ -4 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \right) \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -2 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[20] II.3 Let $\mathbf{C}$ be the matrix

$$\mathbf{C} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ -2 & 1 & -1 \end{bmatrix}$$

(a) Find an orthogonal 3×3 matrix $\mathbf{Q}$ and a matrix $\mathbf{R}$ in echelon form such that $\mathbf{C} = \mathbf{QR}$
.

By observation, the first two columns of $\mathbf{C}$ are orthogonal, and the third column is the sum of the first two (you could get this by doing Gram-Schmidt if you did not immediately see it, as Gram-Schmidt is exactly what you need to get the third column of $\mathbf{Q}$. Normalizing these first two columns gives that

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{2}\sqrt{2} \\ \frac{1}{3} & 0 \\ -\frac{2}{3} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 0 & \sqrt{2} & \sqrt{2} \end{bmatrix} = \mathbf{C}$$

where the last column of the matrix in echelon form just says that 3 times the first column of $\mathbf{Q}$ plus $\sqrt{2}$ times the second column of $\mathbf{Q}$ is equal to the third column of $\mathbf{C}$. But you are asked for an orthogonal, hence square, matrix $\mathbf{Q}$. There are (at least) two ways to do this. One is to find a basis for the left null space, which is the orthogonal complement of the column space of $\mathbf{C}$. One can do this by solving the homogeneous system of equations $\mathbf{C}^T\mathbf{x} = \mathbf{0}$ to get a solution $\mathbf{x} = \begin{bmatrix} 1 & -4 & -1 \end{bmatrix}^T$ or by finding any vector $\mathbf{x}$ such as $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ not in the column space of $\mathbf{C}$ (or $\mathbf{C}$ would have rank 3) and then doing Gram-Schmidt to get the vector

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \right) \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ 0 \\ \frac{1}{2}\sqrt{2} \end{bmatrix} \right) \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ 0 \\ \frac{1}{2}\sqrt{2} \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{2}{9} \\ \frac{1}{9} \\ -\frac{2}{9} \end{bmatrix} - 0 \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ 0 \\ \frac{1}{2}\sqrt{2} \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} \\ \frac{8}{9} \\ \frac{2}{9} \end{bmatrix}$$

which is parallel the vector obtained by looking at the left null space. Normalizing then gives the third column of $\mathbf{Q}$ so

$$\mathbf{Q} = \begin{bmatrix} \frac{2}{3} & \frac{1}{2}\sqrt{2} & -\frac{2}{3\sqrt{72}} \\ \frac{1}{3} & 0 & \frac{8}{3\sqrt{72}} \\ -\frac{2}{3} & \frac{1}{2}\sqrt{2} & \frac{2}{3\sqrt{72}} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{2}\sqrt{2} & -\frac{1}{18}\sqrt{18} \\ \frac{1}{3} & 0 & \frac{2}{9}\sqrt{18} \\ -\frac{2}{3} & \frac{1}{2}\sqrt{2} & \frac{1}{18}\sqrt{18} \end{bmatrix}$$

but now you must make $\mathbf{R}$ 3×3, and that last vector cannot contribute to the produce $\mathbf{QR}$, so set

$$\mathbf{R} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$

and to be on the safe side, one can check

$$\mathbf{C} = \begin{bmatrix} \frac{2}{3} & \frac{1}{2}\sqrt{2} & \frac{1}{18}\sqrt{18} \\ \frac{1}{3} & 0 & -\frac{2}{9}\sqrt{18} \\ -\frac{2}{3} & \frac{1}{2}\sqrt{2} & -\frac{1}{18}\sqrt{18} \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Find the least squares best solution to the inconsistent system of equations

$$\mathbf{C} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The projection of $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ to the column space of $\mathbf{C}$ is $\frac{1}{3} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$ (computed above), so a least squares best fit vector satisfies $\mathbf{C}\bar{\mathbf{x}} = \begin{bmatrix} \frac{2}{9} \\ \frac{1}{9} \\ -\frac{2}{9} \end{bmatrix}$ and an obvious solution to this equation is given by the vector $\bar{\mathbf{x}} = \begin{bmatrix} \frac{1}{9} \\ 0 \\ 0 \end{bmatrix}$.