A Review Sheet for the Final Exam 12/20/2004

Here are some examples of problems you should be able to do on the final. Other examples of problems can be found in the review sheet for the midterm and the answer sheet for the midterm and one of the quizzes, plus from the quizzes themselves. You should also look over your MATLABs. You should also do the review sheet for the final exam from the course given in Fall 2003.

1 Computation

1. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$.

   (a) Find the singular values of $\mathbf{A}$, and a singular value decomposition $\mathbf{A} = \mathbf{U\Sigma V}$ where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices.

   (b) Find the pseudoinverse of $\mathbf{A}$.

   (c) Find an orthogonal matrix $\mathbf{Q}$ and a matrix $\mathbf{S}$ which is symmetric positive semi-definite such that $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} = \mathbf{QS}$.

2. Find the solution of the system of differential equations

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

   satisfying the initial condition $\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

3. Find a matrix $\mathbf{Q}$ with orthonormal columns and a matrix $\mathbf{R}$ in row echelon form such that

$$\mathbf{A} = \mathbf{QR}$$

4. Determine which one of the properties listed below is satisfied by the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

   Is $\mathbf{B}$ positive definite, positive semidefinite, negative definite, negative semidefinite, or none of the other listed options? Justify your answer.

5. Find the least squares best solution of shortest length to the equation

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

2 Theory

1. Indicate whether the following statements are always true, never true, or sometimes true. Justify your answer by giving a brief proof if the statement is always true or always false, or examples illustrating that it is sometimes true and sometimes false if that is the case.

   (a) A 3×3 matrix $\mathbf{A}$ has eigenvalues 1, 2, and -3. Another 3×3 matrix $\mathbf{B}$ has eigenvalues -4, 5, and 6. Then $\mathbf{A}$ and $\mathbf{B}$ must be congruent to each other. (Recall: two matrices $\mathbf{A}$ and $\mathbf{B}$ are congruent to each other if there is a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{B}$.)

   (b) If $\mathbf{A}$ is an $m \times c$ matrix and $\mathbf{x}$ is an $c \times 1$ eigenvector for $\mathbf{A}^H\mathbf{A}$, then $\mathbf{A}x$ is an eigenvector for $\mathbf{A}\mathbf{A}^H$

   (c) If $\mathbf{A}$ is an $n \times n$ complex matrix, and $\mathbf{x}$ is an eigenvector for $\mathbf{A}$ with real eigenvalue λ, then λ is also an eigenvalue for $\mathbf{A}^H$.

   (d) For $\mathbf{A}$ an $m \times n$ matrix with complex number entries, and $\mathbf{x}$ an $m \times 1$ matrix with complex number entries, $\mathbf{A}^H\mathbf{A}\mathbf{x} = \mathbf{0}$ is true if and only if $\mathbf{A}\mathbf{x} = \mathbf{0}$.

   (e) Let $\mathbf{P}$ be a nonsingular $n \times n$ matrix and $\mathbf{D}$ a diagonal $n \times n$ matrix. Set $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. Then $\mathbf{A}$ has some set of n eigenvectors which span $\mathbb{R}^n$.

2. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 4 \\ -1 & 0 & 1 \end{bmatrix}$. Without computing any eigenvalues or eigenvectors of $\mathbf{A}$, determine whether or not there is some set consisting of 3 **pairwise orthogonal** eigenvectors of $\mathbf{A}$ (which must then form a basis of $\mathbb{R}^3$).

3. Let $\mathbf{P}$ be a nonsingular $n \times n$ matrix and $\mathbf{D}$ a diagonal $n \times n$ matrix. Set $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. Find (that is, describe) some set of n eigenvectors of $\mathbf{A}$ which span $\mathbb{R}^n$.