A Review Sheet for the Midterm Oct 27, 2004

Here are some examples of problems you should be able to do on the midterm. I also suggest you look at the review problems used in this course in Fall 2003. They can be accessed by going to the course home page and then to the home page for Fall 2003. Here echelon form means only that all nonzero rows precede all zero rows, and for any row i, the first nonzero entry (if it exists) precedes the first nonzero entry of row $i+1$ (if it exists). There is no assumption that the leading entry is 1, and indeed that is very unlikely.

1 Computation

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

1. Find a permutation matrix $\mathbf{P}$, a lower triangular matrix $\mathbf{L}$ with 1's on the diagonal, and a matrix $\mathbf{U}$ in row echelon form such that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{L}\mathbf{U}$$

2. Use the matrices $\mathbf{L}$ and $\mathbf{U}$ from question 1 directly rather than the matrix $\mathbf{A}$ to solve the following equations or show they have no solutions.

   (a) $\mathbf{LU}\begin{bmatrix} u \\ v \\ w \\ x \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

   (b) $\mathbf{LU}\begin{bmatrix} u \\ v \\ w \\ x \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$

3. Find a matrix $\mathbf{Q}$ with orthonormal columns and a matrix $\mathbf{R}$ in row exhelon form such that

$$\mathbf{A} = \mathbf{QR}$$

4. Find bases for the four fundamental subspaces associated with the matrix $\mathbf{A}$.

5. Find a linear transformation Φ from $\mathbb{R}^4 =$ to the column space of $\mathbf{A}$ such that Φ restricted to the row space of $\mathbf{A}$ maps that row space one-to-one and onto the column space of $\mathbf{A}$.

6. Find the least squares best solution to the equation

$$\mathbf{A}\begin{bmatrix} u \\ v \\ w \\ x \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

7. Find an orthogonal matrix $\tilde{\mathbf{Q}}$ such that the first three columns of $\tilde{\mathbf{Q}}$ are in the column space of $\mathbf{A}$.

2 Theory

Indicate whether the following statements are always true, never true, or sometimes true. Justify your answer by giving a brief proof if the statement is always true or always false, or examples illustrating that it is sometimes true and sometimes false if that is the case.

If a statement is sometimes true and sometimes false, and there is a clearly closely related statement that is always true, give that closely related statement and indicate why it is always true. Note that if a statement is always false, the negation of statement, or the statement that it is always false is an always true statement, so an always false closely related statement can also be given here as long as you indicate that it is always false.

1. (a) A system of linear equations $\mathbf{Ax} = \mathbf{b}$ with more variables than equations must have a solution.

   (b) For $\mathbf{A}$ an $m \times n$ matrix and $\mathbf{B}$ an $n \times p$ matrix and $\mathbf{x}$ a $p \times 1$ matrix, the equation $\mathbf{ABx} = \mathbf{0}$ is true if and only if $\mathbf{Bx} = \mathbf{0}$.

   (c) For $\mathbf{A}$ an $m \times n$ matrix with complex number entries, and $\mathbf{x}$ an $n \times 1$ matrix with complex number entries, $\mathbf{A}^H\mathbf{Ax} = \mathbf{0}$ is true if and only if $\mathbf{Ax} = \mathbf{0}$.

   (d) The orthogonal complement of the x-y plane in euclidean 3-space $\mathbb{R}^3$ is the vector $\hat{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

   (e) Let ω_n be a primitive n^{th} root of unity, and let $F_n[\omega]$ be the Fourier matrix whose p, q entry is ω^{pq} for $0 \leq p, q \leq n-1$. Now set $n = 256$, and set $\omega = \omega_{255}$. The quickest way to solve the system of equations

   $$F_{256}[\omega]\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{255} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ 256 \end{bmatrix}$$

   is to use the fast Fourier tranform to multiply $F_{256}[\omega] \cdot \begin{bmatrix} 1 & 2 & \cdots & 256 \end{bmatrix}^T$.

   (f) To use the fast Fourier transform to multiply $F_{256}[\omega] \cdot \begin{bmatrix} 1 & 2 & \cdots & 256 \end{bmatrix}^T$ we would have to compute the 128 transforms $F_2[-1]\begin{bmatrix} 2k+1 \\ 2k+2 \end{bmatrix}$ for $0 \leq k \leq 127$.