642:550 Applied Linear Algebra
ANSWERS for November 22, 2004 quiz

1. Find all solutions of the differential equation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

All solutions of this differential equation are of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \exp\left(\begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} t \right) \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

where the c_i are constants (so the solution space is 3-dimensional) and

$$\exp(\mathbf{A}t) = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!}$$

We note the basic facts that

$$\begin{aligned} \exp\left(\mathbf{PAP}^{-1}t\right) &= \mathbf{P}\exp(\mathbf{A}t)\,\mathbf{P}^{-1} \qquad \text{for all nonsingular } \mathbf{P}\ ; \\ \mathbf{AB} &= \mathbf{BA} \Longrightarrow \exp((\mathbf{A}+\mathbf{B})\,t) = \exp(\mathbf{A}t) \times \exp(\mathbf{B}t)\ . \end{aligned}$$

This is shown by formal computations using the power series. Similar formal computations show that the columns of $\exp(\mathbf{A}t)$ are solutions of the first order system

$$\mathbf{x}' = \mathbf{Ax}\ .$$

All this assumes that the series converges absolutely. The Schur triangularization procedure can be done to show that every matrix over $\mathbb{C}$ is similar to an upper triangular matrix in which the diagonal elements corresponding to each eigenvalue are grouped together. A procedure shown in class then can be used to make sure that such a matrix is similar to a block diagonal matrix in which every blocks has a single eigenvalue on its diagonal, and all entries outside the blocks are zero. Each of those blocks can then be shown similar to a block diagonal matrix in which each of the blocks is in Jordan Canonical Form, but you do not need that information to compute the exponential of a matrix.

In the question on this quiz, you are given a lower triangular matrix with a single eigenvalue -1. Without any similarity transformation you can find the exponential of such a matrix exactly (modulo not automatically being able to write down the powers of the 'nilpotent' part) the same way you find the exponential of a Jordan block.

Set $\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then

$$\begin{aligned} \mathbf{A}t &= -\mathbf{I}_3 t + \mathbf{N}t \\ (-\mathbf{I}_3 t)(\mathbf{N}t) &= (\mathbf{N}t)(-\mathbf{I}_3 t) = -\mathbf{N}t^2 \end{aligned}$$

so

$$\begin{aligned} \exp(\mathbf{A}t) &= \exp(-\mathbf{I}_3 t)\ \exp(\mathbf{N}t) \\ &= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} & 0 & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} & 0 \\ 0 & 0 & \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2t & 0 & 0 \\ 0 & 1t & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2t & 0 & 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} e^{-t} & 0 & 0 \\ 2te^{-t} & e^{-t} & 0 \\ \frac{2}{2}t^2 e^{-t} & te^{-t} & e^{-t} \end{bmatrix} \end{aligned}$$

and any solution of the given system of differential equations is of the form

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= c_1 \begin{bmatrix} e^{-t} \\ 2te^{-t} \\ \frac{2}{2}t^2 e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{-t} \\ te^{-t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} c_1 e^{-t} \\ 2c_1 te^{-t} + c_2 e^{-t} \\ c_1 t^2 e^{-t} + c_2 te^{-t} + c_3 e^{-t} \end{bmatrix} \end{aligned}$$

We check this.

$$\begin{aligned}
\frac{d}{dt}\left(\begin{bmatrix} c_1e^{-t} \\ 2c_1te^{-t}+c_2e^{-t} \\ c_1t^2e^{-t}+c_2te^{-t}+c_3e^{-t} \end{bmatrix}\right) &= \begin{bmatrix} -c_1e^{-t} \\ 2c_1e^{-t}-c_2e^{-t}-2tc_1e^{-t} \\ c_2e^{-t}-c_3e^{-t}+2tc_1e^{-t}-tc_2e^{-t}-t^2c_1e^{-t} \end{bmatrix} \\
\begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} c_1e^{-t} \\ 2c_1te^{-t}+c_2e^{-t} \\ c_1t^2e^{-t}+c_2te^{-t}+c_3e^{-t} \end{bmatrix} &= \begin{bmatrix} -c_1e^{-t} \\ 2c_1e^{-t}-c_2e^{-t}-2tc_1e^{-t} \\ c_2e^{-t}-c_3e^{-t}+2tc_1e^{-t}-tc_2e^{-t}-t^2c_1e^{-t} \end{bmatrix}
\end{aligned}$$