

## Computing the Jordan Canonical Form

Let  $A$  be an  $n$  by  $n$  square matrix. If its characteristic equation  $\chi_A(t) = 0$  has a repeated root then  $A$  may not be diagonalizable, so we need the Jordan Canonical Form. Suppose  $\lambda$  is an eigenvalue of  $A$ , with multiplicity  $r$  as a root of  $\chi_A(t) = 0$ . The the vector  $\mathbf{v}$  is an eigenvector with eigenvalue  $\lambda$  if  $A\mathbf{v} = \lambda\mathbf{v}$  or equivalently

$$(A - \lambda I)\mathbf{v} = 0.$$

The trouble is that this equation may have fewer than  $r$  linearly independent solutions for  $\mathbf{v}$ . So we generalize and say that  $\mathbf{v}$  is a *generalized eigenvector* with eigenvalue  $\lambda$  if

$$(A - \lambda I)^k \mathbf{v} = 0$$

for some positive integer  $k$ . Now one can prove that there are exactly  $r$  linearly independent generalized eigenvectors. Finding the Jordan form is now a matter of sorting these generalized eigenvectors into an appropriate order.

To find the Jordan form carry out the following procedure for each eigenvalue  $\lambda$  of  $A$ . First solve  $(A - \lambda I)\mathbf{v} = 0$ , counting the number  $r_1$  of linearly independent solutions. If  $r_1 = r$  good, otherwise  $r_1 < r$  and we must now solve  $(A - \lambda I)^2\mathbf{v} = 0$ . There will be  $r_2$  linearly independent solutions where  $r_2 > r_1$ . If  $r_2 = r$  good, otherwise solving  $(A - \lambda I)^3\mathbf{v} = 0$  gives  $r_3 > r_2$  linearly independent solutions, and so on. Eventually one gets  $r_1 < r_2 < \dots < r_{N-1} < r_N = r$ . The number  $N$  is the size of the largest Jordan block associated to  $\lambda$ , and  $r_1$  is the total number of Jordan blocks associated to  $\lambda$ . If we define  $s_1 = r_1$ ,  $s_2 = r_2 - r_1$ ,  $s_3 = r_3 - r_2$ ,  $\dots$ ,  $s_N = r_N - r_{N-1}$  then  $s_k$  is the number of Jordan blocks of size at least  $k$  by  $k$  associated to  $\lambda$ . Finally put  $m_1 = s_1 - s_2$ ,  $m_2 = s_2 - s_3$ ,  $\dots$ ,  $m_{N-1} = s_{N-1} - s_N$  and  $m_N = s_N$ . Then  $m_k$  is the number of  $k$  by  $k$  Jordan blocks associated to  $\lambda$ . Once we've done this for all eigenvalues then we've got the Jordan form!

To find  $P$  such that  $J = P^{-1}AP$  is the Jordan form then we need to work a bit harder. We do the following for each eigenvalue  $\lambda$ . First find the Jordan block sizes associated to  $\lambda$  by the above process. Put them in decreasing order  $N_1 \geq N_2 \geq N_3 \geq \dots \geq N_k$ . Now find a vector  $\mathbf{v}_{1,1}$  such that  $(A - \lambda I)^{N_1}\mathbf{v}_{1,1} = 0$  but  $(A - \lambda I)^{N_1-1}\mathbf{v}_{1,1} \neq 0$ . Define  $\mathbf{v}_{1,2} = (A - \lambda I)\mathbf{v}_{1,1}$ ,  $\mathbf{v}_{1,3} = (A - \lambda I)\mathbf{v}_{1,2}$ , and so on until we get  $\mathbf{v}_{1,N_1}$ . We can't go further as  $(A - \lambda I)\mathbf{v}_{1,N_1} = 0$ . If we only have one block we're OK, otherwise we can find a vector  $\mathbf{v}_{2,1}$  such that  $(A - \lambda I)^{N_2}\mathbf{v}_{2,1} = 0$ ,  $(A - \lambda I)^{N_2-1}\mathbf{v}_{2,1} \neq 0$  and (**this**

**is important!)**  $\mathbf{v}_{2,1}$  is not linearly dependent on  $\mathbf{v}_{1,1}, \dots, \mathbf{v}_{1,N_1}$ . Define  $\mathbf{v}_{2,2} = (A - \lambda I)\mathbf{v}_{2,1}$  etc., until we get to  $\mathbf{v}_{2,N_2}$ . If  $k = 2$  this is the end, if not then choose  $\mathbf{v}_{3,1}$  with  $(A - \lambda I)^{N_3}\mathbf{v}_{3,1} = 0$ ,  $(A - \lambda I)^{N_3-1}\mathbf{v}_{3,1} \neq 0$  and  $\mathbf{v}_{3,1}$  not linearly dependent on  $\mathbf{v}_{1,1}, \dots, \mathbf{v}_{1,N_1}, \mathbf{v}_{2,1}, \dots, \mathbf{v}_{2,N_2}$ . Keep going! Eventually we get  $r$  linearly independent vectors  $\mathbf{v}_{1,1}, \mathbf{v}_{1,2}, \dots, \mathbf{v}_{k,N_k}$ . Let

$$P_\lambda = (\mathbf{v}_{k,N_k} \cdots \mathbf{v}_{1,1})$$

be the  $n$  by  $r$  matrix whose columns are these vectors in **reverse** order. Once we've done this for all eigenvalues  $\lambda$  stick the matrices  $P_\lambda$  together horizontally to get an  $n$  by  $n$  matrix  $P$ . Then  $P$  will be non-singular, and  $P^{-1}AP = J$ , the Jordan form.

### A worked example

To illustrate this method, I give a reasonably sized example (6 by 6) which I hope will make things clear, and I hope is safely too big come up on any exam! I have used MAPLE in the computations; only a truly hardy soul would try this one by hand!

Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & -8 & 4 & -3 & 1 & -3 \\ -3 & 13 & -8 & 6 & 2 & 9 \\ -2 & 14 & -7 & 4 & 2 & 10 \\ 1 & -18 & 11 & -11 & 2 & -6 \\ -1 & 19 & -11 & 10 & -2 & 7 \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$\chi_A(t) = t^6 + 3t^5 - 10t^3 - 15t^2 - 9t - 2 = (t + 1)^5(t - 2)$$

and so its eigenvalues are  $-1$  with multiplicity 5, and 2 with multiplicity 1. I'll deal with  $\lambda = -1$  first. We first solve  $(A + I)\mathbf{v} = 0$ . The matrix

$$A + I = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & -7 & 4 & -3 & 1 & -3 \\ -3 & 13 & -7 & 6 & 2 & 9 \\ -2 & 14 & -7 & 5 & 2 & 10 \\ 1 & -18 & 11 & -11 & 3 & -6 \\ -1 & 19 & -11 & 10 & -2 & 8 \end{pmatrix}$$

has REF

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 3/2 \\ 0 & 0 & 1 & 0 & 2 & 3/2 \\ 0 & 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence  $(A + I)\mathbf{v}$  has 2 linearly independent solutions, i.e.,  $r_1 = 2$ . As  $r_1 < r = 5$  then we must solve  $(A + I)^2\mathbf{v} = 0$ . Now

$$(A + I)^2 = \begin{pmatrix} 1 & -1 & 0 & 1 & -2 & -3 \\ -2 & -16 & 9 & -11 & 4 & -3 \\ -1 & 37 & -18 & 17 & 2 & 21 \\ 1 & 35 & -18 & 19 & -2 & 15 \\ -1 & -53 & 27 & -28 & 2 & -24 \\ 2 & 52 & -27 & 29 & -4 & 21 \end{pmatrix}$$

whose REF is

$$\begin{pmatrix} 1 & 0 & -1/2 & 3/2 & -2 & -5/2 \\ 0 & 1 & -1/2 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The system  $(A + I)^2\mathbf{v}$  has  $r_2 = 4$  linearly independent solutions. As  $r_2 < r$ , then we now consider  $(A + I)^3\mathbf{v}$ . Now

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -54 & 27 & -27 & 0 & -27 \\ 0 & 108 & -54 & 54 & 0 & 54 \\ 0 & 108 & -54 & 54 & 0 & 54 \\ 0 & -162 & 81 & -81 & 0 & -81 \\ 0 & 162 & -81 & 81 & 0 & 81 \end{pmatrix}$$

and it's easy to see (!) that the REF of this matrix is

$$\begin{pmatrix} 0 & 1 & -1/2 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence  $(A + I)^3 \mathbf{v} = 0$  has  $r_3 = 5$  linearly independent solutions, and as  $r_3 = r$  we conclude this part of the proceedings. We calculate  $s_1 = r_1 = 2$ ,  $s_2 = r_2 - r_1 = 2$  and  $s_3 = r_3 - r_2 = 1$ ; also  $m_3 = s_3 = 1$ ,  $m_2 = s_2 - s_3 = 1$  and  $m_1 = s_1 - s_2 = 0$ . Hence, associated to  $\lambda = -1$ , there is a 2 by 2 and a 3 by 3 Jordan block. As the other eigenvalue  $\lambda = 2$  has multiplicity 1, then there's just a 1 by 1 Jordan block associated to  $\lambda = 2$ . Hence the Jordan canonical form of  $A$  is  $J =$

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Let's compute the matrix  $P$ . We've already done most of the work for  $\lambda = -1$ . The Jordan blocks have sizes  $N_1 = 3$  and  $N_2 = 2$ . We start by finding a vector  $\mathbf{v}_{1,1}$  with  $(A + I)^3 \mathbf{v}_{1,1} = 0$  but  $(A + I)^2 \mathbf{v}_{1,1} \neq 0$ . Looking at the REFs of these matrices we see that we can choose

$$\mathbf{v}_{1,1} = (1 \ 0 \ 0 \ 0 \ 0 \ 0)^t.$$

Now

$$\mathbf{v}_{1,2} = (A + I)\mathbf{v}_{1,1} = (1 \ 0 \ -3 \ -2 \ 1 \ -1)^t$$

and

$$\mathbf{v}_{1,3} = (A + I)\mathbf{v}_{1,2} = (1 \ -2 \ -1 \ 1 \ -1 \ 2)^t.$$

(As a check one verifies  $(A + I)\mathbf{v}_{1,3} = 0$ .) The next block is 2 by 2, so one must find  $\mathbf{v}_{2,1}$  with  $(A + I)^2 \mathbf{v}_{2,1} = 0$ ,  $(A + I)\mathbf{v}_{2,1} \neq 0$ , and such that  $\mathbf{v}_{2,1}$  is not linearly dependent on  $\mathbf{v}_{1,1}$ ,  $\mathbf{v}_{1,2}$  and  $\mathbf{v}_{1,3}$ . The vector

$$\mathbf{v}_{2,1} = (1 \ 1 \ 2 \ 0 \ 0 \ 0)^t$$

fits the bill, and

$$\mathbf{v}_{2,2} = (A + I)\mathbf{v}_{2,1} = (1 \ 1 \ -4 \ -2 \ 5 \ -4)^t.$$

Again one checks that  $(A + I)\mathbf{v}_{2,2} = 0$ . The matrix  $P_{-1}$  is the 6 by 5 matrix with columns  $\mathbf{v}_{2,2}$ ,  $\mathbf{v}_{2,1}$ ,  $\mathbf{v}_{1,3}$ ,  $\mathbf{v}_{1,2}$  and  $\mathbf{v}_{1,1}$  in that order and so

$$P_{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & 0 \\ -4 & 2 & -1 & -3 & 0 \\ -2 & 0 & 1 & -2 & 0 \\ 5 & 0 & -1 & 1 & 0 \\ -4 & 0 & 2 & -1 & 0 \end{pmatrix}.$$

One must now consider  $\lambda = 2$ . As this is a simple root,  $P_2$  is just an eigenvector with eigenvalue 2. One such is

$$P_2 = (0 \ 1 \ -2 \ -2 \ 3 \ -3)^t$$

and sticking together  $P_{-1}$  and  $P_2$  gives

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \\ -4 & 2 & -1 & -3 & 0 & -2 \\ -2 & 0 & 1 & -2 & 0 & -2 \\ 5 & 0 & -1 & 1 & 0 & 3 \\ -4 & 0 & 2 & -1 & 0 & -3 \end{pmatrix}.$$

One now checks that  $P^{-1}AP = J$  as required!

RJC 25/1/95