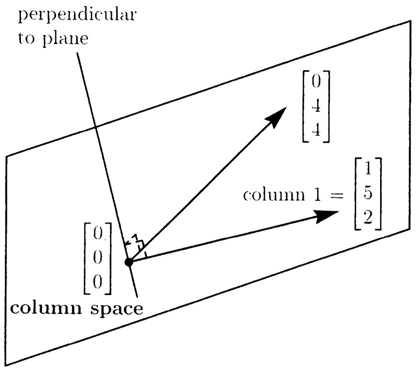

Figure 2.1 The column space $C(A)$, a plane in three-dimensional space.

$C(A)$. Requirements (i) and (ii) for a subspace of $\mathbf{R}^m$ are easy to check:

(i) Suppose b and b' lie in the column space, so that $Ax = b$ for some x and $Ax' = b'$ for some x'. Then $A(x + x') = b + b'$, so that $b + b'$ is also a combination of the columns. The column space of all attainable vectors b is closed under addition.

(ii) If b is in the column space $C(A)$, so is any multiple cb. If some combination of columns produces b (say $Ax = b$), then multiplying that combination by c will produce cb. In other words, $A(cx) = cb$.

For another matrix A, the dimensions in Figure 2.1 may be very different. The smallest possible column space (one vector only) comes from the zero matrix $A = 0$. The only combination of the columns is $b = 0$. At the other extreme, suppose A is the 5 by 5 identity matrix. Then $C(I)$ is the whole of $\mathbf{R}^5$; the five columns of I can combine to produce any five-dimensional vector b. This is not at all special to the identity matrix. *Any 5 by 5 matrix that is nonsingular will have the whole of* $\mathbf{R}^5$ *as its column space.* For such a matrix we can solve $Ax = b$ by Gaussian elimination; there are five pivots. Therefore every b is in $C(A)$ for a nonsingular matrix.

You can see how Chapter 1 is contained in this chapter. There we studied n by n matrices whose column space is $\mathbf{R}^n$. Now we allow singular matrices, and rectangular matrices of any shape. Then $C(A)$ can be somewhere between the zero space and the whole space $\mathbf{R}^m$. Together with its perpendicular space, it gives one of our two approaches to understanding $Ax = b$.

The Nullspace of A

The second approach to $Ax = b$ is "dual" to the first. We are concerned not only with attainable right-hand sides b, but also with the solutions x that attain them. The right-hand side $b = 0$ always allows the solution $x = 0$, but there may be infinitely many other solutions. (There always are, if there are more unknowns than equations, $n > m$.) ***The solutions to $Ax = 0$ form a vector space—the nullspace of A.***

The ***nullspace*** of a matrix consists of all vectors x such that $Ax = 0$. It is denoted by $N(A)$. It is a subspace of $\mathbf{R}^n$, just as the column space was a subspace of $\mathbf{R}^m$.

Requirement (i) holds: If $Ax = 0$ and $Ax' = 0$, then $A(x + x') = 0$. Requirement (ii) also holds: If $Ax = 0$ then $A(cx) = 0$. Both requirements fail if the right-hand side is not zero! Only the solutions to a *homogeneous* equation ($b = 0$) form a subspace. The nullspace is easy to find for the example given above; it is as small as possible:

$$\begin{bmatrix}1&0\\5&4\\2&4\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The first equation gives $u = 0$, and the second equation then forces $v = 0$. The nullspace contains only the vector $(0, 0)$. This matrix has "independent columns"—a key idea that comes soon.

The situation is changed when a third column is a combination of the first two:

$$\textbf{Larger nullspace}\qquad B=\begin{bmatrix}1&0&1\\5&4&9\\2&4&6\end{bmatrix}.$$

B has the same column space as A. The new column lies in the plane of Figure 2.1; it is the sum of the two column vectors we started with. But the nullspace of B contains the vector $(1, 1, -1)$ and automatically contains any multiple $(c, c, -c)$:

$$\textbf{Nullspace is a line}\qquad \begin{bmatrix}1&0&1\\5&4&9\\2&4&6\end{bmatrix}\begin{bmatrix}c\\c\\-c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The nullspace of B is the line of all points $x = c$, $y = c$, $z = -c$. (The line goes through the origin, as any subspace must.) We want to be able, for any system $Ax = b$, to find $C(A)$ and $N(A)$: all attainable right-hand sides b and all solutions to $Ax = 0$.

The vectors b are in the column space and the vectors x are in the nullspace. We shall compute the dimensions of those subspaces and a convenient set of vectors to generate them. We hope to end up by understanding all *four* of the subspaces that are intimately related to each other and to A—the column space of A, the nullspace of A, and their two perpendicular spaces.

Problem Set 2.1

1. Construct a subset of the x-y plane $\mathbf{R}^2$ that is

(a) closed under vector addition and subtraction, but not scalar multiplication.

(b) closed under scalar multiplication but not under vector addition.

Hint: Starting with u and v, add and subtract for (a). Try cu and cv for (b).

2. Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0$.

(b) The plane of vectors b with $b_1 = 1$.
(c) The vectors b with $b_2 b_3 = 0$ (this is the union of two subspaces, the plane $b_2 = 0$ and the plane $b_3 = 0$).
(d) All combinations of two given vectors $(1, 1, 0)$ and $(2, 0, 1)$.
(e) The plane of vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0$.

3. Describe the column space and the nullspace of the matrices

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

4. What is the smallest subspace of 3 by 3 matrices that contains all symmetric matrices *and* all lower triangular matrices? What is the largest subspace that is contained in both of those subspaces?

5. Addition and scalar multiplication are required to satisfy these eight rules:

1. $x + y = y + x$.
2. $x + (y + z) = (x + y) + z$.
3. There is a unique "zero vector" such that $x + 0 = x$ for all x.
4. For each x there is a unique vector $-x$ such that $x + (-x) = 0$.
5. $1x = x$.
6. $(c_1 c_2)x = c_1(c_2 x)$.
7. $c(x + y) = cx + cy$.
8. $(c_1 + c_2)x = c_1 x + c_2 x$.

(a) Suppose addition in $\mathbf{R}^2$ adds an extra 1 to each component, so that $(3, 1) + (5, 0)$ equals $(9, 2)$ instead of $(8, 1)$. With scalar multiplication unchanged, which rules are broken?
(b) Show that the set of all positive real numbers, with $x + y$ and cx redefined to equal the usual xy and x^c, is a vector space. What is the "zero vector"?
(c) Suppose $(x_1, x_2) + (y_1, y_2)$ is defined to be $(x_1 + y_2, x_2 + y_1)$. With the usual $cx = (cx_1, cx_2)$, which of the eight conditions are not satisfied?

6. Let $\mathbf{P}$ be the plane in 3-space with equation $x + 2y + z = 6$. What is the equation of the plane $\mathbf{P}_0$ through the origin parallel to $\mathbf{P}$? Are $\mathbf{P}$ and $\mathbf{P}_0$ subspaces of $\mathbf{R}^3$?

7. Which of the following are subspaces of $\mathbf{R}^\infty$?
(a) All sequences like $(1, 0, 1, 0, \ldots)$ that include infinitely many zeros.
(b) All sequences $(x_1, x_2, \ldots)$ with $x_j = 0$ from some point onward.
(c) All decreasing sequences: $x_{j+1} \le x_j$ for each j.
(d) All convergent sequences: the x_j have a limit as $j \to \infty$.
(e) All arithmetic progressions: $x_{j+1} - x_j$ is the same for all j.
(f) All geometric progressions $(x_1, kx_1, k^2 x_1, \ldots)$ allowing all k and x_1.

8. Which of the following descriptions are correct? The solutions x of

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

form
(a) a plane.
(b) a line.
(c) a point.
(d) a subspace.
(e) the nullspace of A.
(f) the column space of A.

9. Show that the set of nonsingular 2 by 2 matrices is not a vector space. Show also that the set of *singular* 2 by 2 matrices is not a vector space.

10. The matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is a "vector" in the space $\mathbf{M}$ of all 2 by 2 matrices. Write the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$. What matrices are in the smallest subspace containing A?

11. (a) Describe a subspace of $\mathbf{M}$ that contains $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but not $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.
(b) If a subspace of $\mathbf{M}$ contains A and B, must it contain I?
(c) Describe a subspace of $\mathbf{M}$ that contains no nonzero diagonal matrices.

12. The functions $f(x) = x^2$ and $g(x) = 5x$ are "vectors" in the vector space $\mathbf{F}$ of all real functions. The combination $3f(x) - 4g(x)$ is the function $h(x) =$ ______. Which rule is broken if multiplying $f(x)$ by c gives the function $f(cx)$?

13. If the sum of the "vectors" $f(x)$ and $g(x)$ in $\mathbf{F}$ is defined to be $f(g(x))$, then the "zero vector" is $g(x) = x$. Keep the usual scalar multiplication $cf(x)$, and find two rules that are broken.

14. Describe the smallest subspace of the 2 by 2 matrix space $\mathbf{M}$ that contains

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. (d) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

15. Let $\mathbf{P}$ be the plane in $\mathbf{R}^3$ with equation $x + y - 2z = 4$. The origin $(0, 0, 0)$ is not in $\mathbf{P}$! Find two vectors in $\mathbf{P}$ and check that their sum is not in $\mathbf{P}$.

16. $\mathbf{P}_0$ is the plane through $(0, 0, 0)$ parallel to the plane $\mathbf{P}$ in Problem 15. What is the equation for $\mathbf{P}_0$? Find two vectors in $\mathbf{P}_0$ and check that their sum is in $\mathbf{P}_0$.

17. The four types of subspaces of $\mathbf{R}^3$ are planes, lines, $\mathbf{R}^3$ itself, or $\mathbf{Z}$ containing only $(0, 0, 0)$.
(a) Describe the three types of subspaces of $\mathbf{R}^2$.
(b) Describe the five types of subspaces of $\mathbf{R}^4$.

18. (a) The intersection of two planes through $(0, 0, 0)$ is probably a ______ but it could be a ______. It can't be the zero vector $\mathbf{Z}$!
(b) The intersection of a plane through $(0, 0, 0)$ with a line through $(0, 0, 0)$ is probably a ______ but it could be a ______.
(c) If $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{R}^5$, their intersection $\mathbf{S} \cap \mathbf{T}$ (vectors in both subspaces) is a subspace of $\mathbf{R}^5$. *Check the requirements on $x + y$ and cx.*

19. Suppose **P** is a plane through (0, 0, 0) and **L** is a line through (0, 0, 0). The smallest vector space containing both **P** and **L** is either ______ or ______.

20. True or false for $\mathbf{M}$ = all 3 by 3 matrices (check addition using an example)?

(a) The skew-symmetric matrices in $\mathbf{M}$ (with $A^T = -A$) form a subspace.
(b) The unsymmetric matrices in $\mathbf{M}$ (with $A^T \neq A$) form a subspace.
(c) The matrices that have (1, 1, 1) in their nullspace form a subspace.

Problems 21–30 are about column spaces $C(A)$ and the equation $Ax = b$.

21. Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

22. For which right-hand sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$\text{(a)} \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \qquad \text{(b)} \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

23. Adding row 1 of A to row 2 produces B. Adding column 1 to column 2 produces C. A combination of the columns of ______ is also a combination of the columns of A. Which two matrices have the same column ______?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

24. For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

25. (Recommended) If we add an extra column b to a matrix A, then the column space gets larger unless ______. Give an example in which the column space gets larger and an example in which it doesn't. Why is $Ax = b$ solvable exactly when the column space *doesn't* get larger by including b?

26. The columns of AB are combinations of the columns of A. This means: *The column space of AB is contained in* (possibly equal to) *the column space of A*. Give an example where the column spaces of A and AB are not equal.

27. If A is any 8 by 8 invertible matrix, then its column space is ______. Why?

28. True or false (with a counterexample if false)?

(a) The vectors b that are not in the column space $C(A)$ form a subspace.
(b) If $C(A)$ contains only the zero vector, then A is the zero matrix.
(c) The column space of $2A$ equals the column space of A.
(d) The column space of $A - I$ equals the column space of A.

29. Construct a 3 by 3 matrix whose column space contains (1, 1, 0) and (1, 0, 1) but not (1, 1, 1). Construct a 3 by 3 matrix whose column space is only a line.

30. If the 9 by 12 system $Ax = b$ is solvable for every b, then $C(A)$ = ______.

31. Why isn't $\mathbf{R}^2$ a subspace of $\mathbf{R}^3$?

2.2 SOLVING $Ax = 0$ AND $Ax = b$

Chapter 1 concentrated on square invertible matrices. There was one solution to $Ax = b$, and it was $x = A^{-1}b$. That solution was found by elimination (not by computing A^{-1}). A rectangular matrix brings new possibilities—U may not have a full set of pivots. This section goes onward from U to a reduced form R—**the simplest matrix that elimination can give**. R reveals all solutions immediately.

For an invertible matrix, the nullspace contains only $x = 0$ (multiply $Ax = 0$ by A^{-1}). The column space is the whole space ($Ax = b$ has a solution for every b). The new questions appear when the nullspace contains *more than the zero vector* and/or the column space contains *less than all vectors*:

1. Any vector x_n in the nullspace can be added to a particular solution x_p. The solutions to all linear equations have this form, $x = x_p + x_n$:

   Complete solution $\quad Ax_p = b$ **and** $Ax_n = 0$ **produce** $A(x_p + x_n) = b$.

2. When the column space doesn't contain every b in $\mathbf{R}^m$, we need the conditions on b that make $Ax = b$ solvable.

A 3 by 4 example will be a good size. We will write down all solutions to $Ax = 0$. We will find the conditions for b to lie in the column space (so that $Ax = b$ is solvable). The 1 by 1 system $0x = b$, one equation and one unknown, shows two possibilities:

$0x = b$ has *no solution* unless $b = 0$. The column space of the 1 by 1 zero matrix contains only $b = 0$.

$0x = 0$ has *infinitely many solutions*. The nullspace contains *all* x. A particular solution is $x_p = 0$, and the complete solution is $x = x_p + x_n = 0 +$ (any x).

Simple, I admit. If you move up to 2 by 2, it's more interesting. The matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not invertible: $y + z = b_1$ and $2y + 2z = b_2$ usually have no solution.

There is ***no solution*** unless $b_2 = 2b_1$. The column space of A contains only those b's, the multiples of (1, 2).

When $b_2 = 2b_1$ there are ***infinitely many solutions***. A particular solution to $y + z = 2$ and $2y + 2z = 4$ is $x_p = (1, 1)$. The nullspace of A in Figure 2.2 contains $(-1, 1)$ and all its multiples $x_n = (-c, c)$:

$$\textbf{Complete solution} \quad \begin{matrix} y + z = 2 \\ 2y + 2z = 4 \end{matrix} \quad \text{is solved by} \quad x_p + x_n = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - c \\ 1 + c \end{bmatrix}.$$

Another Worked Example

The full picture uses elimination and pivot columns to find the column space, nullspace, and rank. The 3 by 4 matrix A has rank 2:

$$Ax = b \quad \text{is} \quad \begin{aligned} \mathbf{1}x_1 + \mathbf{2}x_2 + \mathbf{3}x_3 + \ \mathbf{5}x_4 &= b_1 \\ \mathbf{2}x_1 + \mathbf{4}x_2 + \mathbf{8}x_3 + \mathbf{12}x_4 &= b_2 \\ \mathbf{3}x_1 + \mathbf{6}x_2 + \mathbf{7}x_3 + \mathbf{13}x_4 &= b_3 \end{aligned} \tag{6}$$

1. Reduce $[A \ \ b]$ to $[U \ \ c]$, to reach a triangular system $Ux = c$.
2. Find the condition on b_1, b_2, b_3 to have a solution.
3. Describe the column space of A: Which plane in $\mathbf{R}^3$?
4. Describe the nullspace of A: Which special solutions in $\mathbf{R}^4$?
5. Find a particular solution to $Ax = (0, 6, -6)$ and the complete $x_p + x_n$.
6. Reduce $[U \ \ c]$ to $[R \ \ d]$: Special solutions from R and x_p from d.

Solution (Notice how the right-hand side is included as an extra column!)

1. The multipliers in elimination are 2 and 3 and -1, taking $[A \ \ b]$ to $[U \ \ c]$.

$$\begin{bmatrix} 1 & 2 & 3 & 5 & \mathbf{b_1} \\ 2 & 4 & 8 & 12 & \mathbf{b_2} \\ 3 & 6 & 7 & 13 & \mathbf{b_3} \end{bmatrix} \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & \mathbf{b_1} \\ 0 & 0 & 2 & 2 & \mathbf{b_2 - 2b_1} \\ 0 & 0 & -2 & -2 & \mathbf{b_3 - 3b_1} \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & \mathbf{b_1} \\ 0 & 0 & 2 & 2 & \mathbf{b_2 - 2b_1} \\ 0 & 0 & 0 & 0 & \mathbf{b_3 + b_2 - 5b_1} \end{array}\right].$$

2. The last equation shows the solvability condition $b_3 + b_2 - 5b_1 = 0$. Then $0 = 0$.
3. The column space of A is the plane containing all combinations of the pivot columns $(1, 2, 3)$ and $(3, 8, 7)$. **Second description**: The column space contains all vectors with $b_3 + b_2 - 5b_1 = 0$. That makes $Ax = b$ solvable, so b is in the column space. *All columns of A pass this test $b_3 + b_2 - 5b_1 = 0$. This is the equation for the plane (in the first description of the column space).*
4. The special solutions in N have free variables $x_2 = 1, x_4 = 0$ and $x_2 = 0, x_4 = 1$:

Nullspace matrix
Special solutions to $Ax = 0$
Back-substitution in $Ux = 0$
Just switch signs in $Rx = 0$

$$N = \begin{bmatrix} -2 & -2 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

5. Choose $b = (0, 6, -6)$, which has $b_3 + b_2 - 5b_1 = 0$. Elimination takes $Ax = b$ to $Ux = c = (0, 6, 0)$. Back-substitute with free variables $= 0$:

Particular solution to $Ax_p = (0, 6, -6)$ $\quad x_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} \begin{matrix} \\ \text{free} \\ \\ \text{free} \end{matrix}$

The complete solution to $Ax = (0, 6, -6)$ is (this x_p) + (all x_n).

6. In the reduced R, the third column changes from $(3, 2, 0)$ to $(0, 1, 0)$. The right-hand side $c = (0, 6, 0)$ becomes $d = (-9, 3, 0)$. Then -9 and 3 go into x_p:

$$\begin{bmatrix} U & c \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} R & d \end{bmatrix} = \left[\begin{array}{cccc|c} \mathbf{1} & 2 & 0 & 2 & \mathbf{-9} \\ 0 & 0 & \mathbf{1} & 1 & \mathbf{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

That final matrix $[R \ \ d]$ is rref($[A \ \ b]$) = rref($[U \ \ c]$). The numbers 2 and 0 and 2 and 1 in the free columns of R have opposite sign in the special solutions (the nullspace matrix N). Everything is revealed by $Rx = d$.

Problem Set 2.2

1. Construct a system with more unknowns than equations, but no solution. Change the right-hand side to zero and find all solutions x_n.

2. Reduce A and B to echelon form, to find their ranks. Which variables are free?

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Find the special solutions to $Ax = 0$ and $Bx = 0$. Find all solutions.

3. Find the echelon form U, the free variables, and the special solutions:

$$A = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

$Ax = b$ is consistent (has a solution) when b satisfies $b_2 =$ ______. Find the complete solution in the same form as equation (4).

4. Carry out the same steps as in the previous problem to find the complete solution of $Mx = b$:

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

5. Write the complete solutions $x = x_p + x_n$ to these systems, as in equation (4):

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

6. Describe the set of attainable right-hand sides b (in the column space) for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

by finding the constraints on b that turn the third equation into $0 = 0$ (after elimination). What is the rank, and a particular solution?

7. Find the value of c that makes it possible to solve $Ax = b$, and solve it:

$$\begin{aligned} u + \ v + 2w &= 2 \\ 2u + 3v - \ w &= 5 \\ 3u + 4v + \ w &= c. \end{aligned}$$

8. Under what conditions on b_1 and b_2 (if any) does $Ax = b$ have a solution?

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Find two vectors in the nullspace of A, and the complete solution to $Ax = b$.

9. (a) Find the special solutions to $Ux = 0$. Reduce U to R and repeat:

$$Ux = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b) If the right-hand side is changed from $(0, 0, 0)$ to $(a, b, 0)$, what are all solutions?

10. Find a 2 by 3 system $Ax = b$ whose complete solution is

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

Find a 3 by 3 system with these solutions exactly when $b_1 + b_2 = b_3$.

11. Write a 2 by 2 system $Ax = b$ with many solutions x_n but no solution x_p. (Therefore the system has no solution.) Which b's allow an x_p?

12. Which of these rules give a correct definition of the *rank* of A?

(a) The number of nonzero rows in R.
(b) The number of columns minus the total number of rows.
(c) The number of columns minus the number of free columns.
(d) The number of 1s in R.

13. Find the reduced row echelon forms R and the rank of these matrices:

(a) The 3 by 4 matrix of all 1s.
(b) The 4 by 4 matrix with $a_{ij} = (-1)^{ij}$.
(c) The 3 by 4 matrix with $a_{ij} = (-1)^j$.

14. Find R for each of these (block) matrices, and the special solutions:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} A & A \end{bmatrix} \qquad C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}.$$

15. If the r pivot variables come first, the reduced R must look like

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \qquad \begin{matrix} I \text{ is } r \text{ by } r \\ F \text{ is } r \text{ by } n - r \end{matrix}$$

What is the nullspace matrix N containing the special solutions?

16. Suppose all r pivot variables come *last*. Describe the four blocks in the m by n reduced echelon form (the block B should be r by r):

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

What is the nullspace matrix N of special solutions? What is its shape?

17. (Silly problem) Describe all 2 by 3 matrices A_1 and A_2 with row echelon forms R_1 and R_2, such that $R_1 + R_2$ is the row echelon form of $A_1 + A_2$. Is it true that $R_1 = A_1$ and $R_2 = A_2$ in this case?

18. If A has r pivot columns, then A^T has r pivot columns. Give a 3 by 3 example for which the column numbers are different for A and A^T.

19. What are the special solutions to $Rx = 0$ and $R^T y = 0$ for these R?

$$R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

20. *If A has rank r, then it has an r by r submatrix S that is invertible.* Find that submatrix S from the pivot rows and pivot columns of each A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

21. Explain why the pivot rows and pivot columns of A (not R) always give an r by r invertible submatrix of A.

22. Find the ranks of AB and AM (rank 1 matrix times rank 1 matrix):

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1.5 & 6 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & b \\ c & bc \end{bmatrix}.$$

23. Multiplying the rank 1 matrices $A = uv^T$ and $B = wz^T$ gives uz^T times the number ___. AB has rank 1 unless ___ $= 0$.

24. Every column of AB is a combination of the columns of A. Then the dimensions of the column spaces give **rank(AB) $\leq$ rank(A)**. Problem: Prove also that **rank(AB) $\leq$ rank(B)**.

25. (Important) Suppose A and B are n by n matrices, and $AB = I$. Prove from rank$(AB) \leq$ rank(A) that the rank of A is n. So A is invertible and B must be its two-sided inverse. Therefore $BA = I$ (*which is not so obvious!*).

26. If A is 2 by 3 and C is 3 by 2, show from its rank that $CA \neq I$. Give an example in which $AC = I$. For $m < n$, a right inverse is not a left inverse.

27. Suppose A and B have the *same* reduced-row echelon form R. Explain how to change A to B by elementary row operations. So B equals an ______ matrix times A.

28. Every m by n matrix of rank r reduces to (m by r) times (r by n):

$$A = (\text{pivot columns of } A)(\text{first } r \text{ rows of } R) = (\textbf{COL})(\textbf{ROW}).$$

Write the 3 by 4 matrix A at the start of this section as the product of the 3 by 2 matrix from the pivot columns and the 2 by 4 matrix from R:

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

29. Suppose A is an m by n matrix of rank r. Its reduced echelon form is R. Describe exactly the *reduced row echelon form of* R^T (not A^T).

30. (Recommended) Execute the six steps following equation (6) to find the column space and nullspace of A and the solution to $Ax = b$:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}.$$

31. For every c, find R and the special solutions to $Ax = 0$:

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}.$$

32. What is the nullspace matrix N (of special solutions) for A, B, C?

$$A = \begin{bmatrix} I & I \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} I & I & I \end{bmatrix}.$$

Problems 33–36 are about the solution of $Ax = b$. Follow the steps in the text to x_p and x_n. Reduce the augmented matrix $[A \ \ b]$.

33. Find the complete solutions of

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5 \end{aligned} \quad \text{and} \quad \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

34. Under what condition on b_1, b_2, b_3 is the following system solvable? Include b as a fourth column in $[A \ \ b]$. Find all solutions when that condition holds:

$$\begin{aligned} x + 2y - 2z &= b_1 \\ 2x + 5y - 4z &= b_2 \\ 4x + 9y - 8z &= b_3. \end{aligned}$$

35. What conditions on b_1, b_2, b_3, b_4 make each system solvable? *Solve for* x:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

36. Which vectors (b_1, b_2, b_3) are in the column space of A? Which combinations of the rows of A give zero?

$$\text{(a) } A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix} \qquad \text{(b) } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}.$$

37. Why can't a 1 by 3 system have $x_p = (2, 4, 0)$ and $x_n =$ any multiple of $(1, 1, 1)$?

38. (a) If $Ax = b$ has two solutions x_1 and x_2, find two solutions to $Ax = 0$.
(b) Then find another solution to $Ax = b$.

39. Explain why all these statements are false:
(a) The complete solution is any linear combination of x_p and x_n.
(b) A system $Ax = b$ has at most one particular solution.
(c) The solution x_p with all free variables zero is the shortest solution (minimum length $\|x\|$). (Find a 2 by 2 counterexample.)
(d) If A is invertible there is no solution x_n in the nullspace.

40. Suppose column 5 of U has no pivot. Then x_5 is a ______ variable. The zero vector (is) (is not) the only solution to $Ax = 0$. If $Ax = b$ has a solution, then it has ______ solutions.

41. If you know x_p (free variables $= 0$) and all special solutions for $Ax = b$, find x_p and all special solutions for these systems:

$$Ax = 2b \qquad \begin{bmatrix} A & A \end{bmatrix} \begin{bmatrix} x \\ X \end{bmatrix} = b \qquad \begin{bmatrix} A \\ A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix}.$$

42. If $Ax = b$ has infinitely many solutions, why is it impossible for $Ax = B$ (new right-hand side) to have only one solution? Could $Ax = B$ have no solution?

43. Choose the number q so that (if possible) the ranks are (a) 1, (b) 2, (c) 3:

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}.$$

44. Give examples of matrices A for which the number of solutions to $Ax = b$ is
(a) 0 or 1, depending on b.
(b) ∞, regardless of b.
(c) 0 or ∞, depending on b.
(d) 1, regardless of b.

45. Write all known relations between r and m and n if $Ax = b$ has
(a) no solution for some b.
(b) infinitely many solutions for every b.
(c) exactly one solution for some b, no solution for other b.
(d) exactly one solution for every b.

46. Apply Gauss–Jordan elimination (right-hand side becomes extra column) to $Ux = 0$ and $Ux = c$. Reach $Rx = 0$ and $Rx = d$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix}.$$

Solve $Rx = 0$ to find x_n (its free variable is $x_2 = 1$). Solve $Rx = d$ to find x_p (its free variable is $x_2 = 0$).

47. Apply elimination with the extra column to reach $Rx = 0$ and $Rx = d$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & \mathbf{0} \\ 0 & 0 & 2 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & \mathbf{9} \\ 0 & 0 & 2 & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}.$$

Solve $Rx = 0$ (free variable = 1). What are the solutions to $Rx = d$?

48. Reduce to $Ux = c$ (Gaussian elimination) and then $Rx = d$:

$$Ax = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = b.$$

Find a particular solution x_p and all nullspace solutions x_n.

49. Find A and B with the given property or explain why you can't.

(a) The only solution to $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) The only solution to $Bx = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

50. The complete solution to $Ax = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find A.

51. The nullspace of a 3 by 4 matrix A is the line through (2, 3, 1, 0).

(a) What is the *rank* of A and the complete solution to $Ax = 0$?
(b) What is the exact row reduced echelon form R of A?

52. Reduce these matrices A and B to their ordinary echelon forms U:

$$\text{(a)}\ A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \qquad \text{(b)}\ B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Find a special solution for each free variable and describe every solution to $Ax = 0$ and $Bx = 0$. Reduce the echelon forms U to R, and draw a box around the identity matrix in the pivot rows and pivot columns.

53. True or False? (Give reason if true, or counterexample to show it is false.)

(a) A square matrix has no free variables.
(b) An invertible matrix has no free variables.
(c) An m by n matrix has no more than n pivot variables.
(d) An m by n matrix has no more than m pivot variables.

54. Is there a 3 by 3 matrix with no zero entries for which $U = R = I$?

55. Put as many 1s as possible in a 4 by 7 echelon matrix U and in a *reduced* form R whose pivot columns are 2, 4, 5.

56. Suppose column 4 of a 3 by 5 matrix is all 0s. Then x_4 is certainly a ______ variable. The special solution for this variable is the vector $x =$ ______.

57. Suppose the first and last columns of a 3 by 5 matrix are the same (nonzero). Then ______ is a free variable. Find the special solution for this variable.

58. The equation $x - 3y - z = 0$ determines a plane in $\mathbf{R}^3$. What is the matrix A in this equation? Which are the free variables? The special solutions are (3, 1, 0) and ______. The parallel plane $x - 3y - z = 12$ contains the particular point (12, 0, 0). All points on this plane have the following form (fill in the first components):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \ \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} \ \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} \ \\ 0 \\ 1 \end{bmatrix}.$$

59. Suppose column 1 + column 3 + column 5 = 0 in a 4 by 5 matrix with four pivots. Which column is sure to have no pivot (and which variable is free)? What is the special solution? What is the nullspace?

Problems 60–66 ask for matrices (if possible) with specific properties.

60. Construct a matrix whose nullspace consists of all combinations of (2, 2, 1, 0) and (3, 1, 0, 1).

61. Construct a matrix whose nullspace consists of all multiples of (4, 3, 2, 1).

62. Construct a matrix whose column space contains (1, 1, 5) and (0, 3, 1) and whose nullspace contains (1, 1, 2).

63. Construct a matrix whose column space contains (1, 1, 0) and (0, 1, 1) and whose nullspace contains (1, 0, 1) and (0, 0, 1).

64. Construct a matrix whose column space contains (1, 1, 1) and whose nullspace is the line of multiples of (1, 1, 1, 1).

65. Construct a 2 by 2 matrix whose nullspace equals its column space.

66. Why does no 3 by 3 matrix have a nullspace that equals its column space?

67. The reduced form R of a 3 by 3 matrix with randomly chosen entries is almost sure to be ______. What R is virtually certain if the random A is 4 by 3?

68. Show by example that these three statements are generally *false*:

(a) A and A^{T} have the same nullspace.
(b) A and A^{T} have the same free variables.
(c) If R is the reduced form rref(A) then R^{T} is rref(A^{T}).

69. If the special solutions to $Rx = 0$ are in the columns of these N, go backward to find the nonzero rows of the reduced matrices R:

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \ \end{bmatrix} \quad \text{(empty 3 by 1)}.$$

70. Explain why A and $-A$ always have the same reduced echelon form R.

four-dimensional ***subspace***; an example is the set of vectors in $\mathbf{R}^6$ whose first and last components are zero. The members of this four-dimensional subspace are six-dimensional vectors like $(0, 5, 1, 3, 4, 0)$.

One final note about the language of linear algebra. We never use the terms "basis of a matrix" or "rank of a space" or "dimension of a basis." These phrases have no meaning. It is *the dimension of the column space* that equals *the rank of the matrix*, as we prove in the coming section.

Problem Set 2.3

Problems 1–10 are about linear independence and linear dependence.

1. Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:

$$v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} \quad v_4 = \begin{bmatrix}2\\3\\4\end{bmatrix}.$$

Solve $c_1v_1 + \cdots + c_4v_4 = 0$ or $Ac = 0$. The v's go in the columns of A.

2. Find the largest possible number of independent vectors among

$$v_1 = \begin{bmatrix}1\\-1\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\0\\-1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\0\\0\\-1\end{bmatrix} \quad v_4 = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix} \quad v_5 = \begin{bmatrix}0\\1\\0\\-1\end{bmatrix} \quad v_6 = \begin{bmatrix}0\\0\\1\\-1\end{bmatrix}.$$

This number is the ______ of the space spanned by the v's.

3. Prove that if $a = 0$, $d = 0$, or $f = 0$ (3 cases), the columns of U are dependent:

$$U = \begin{bmatrix}a & b & c\\0 & d & e\\0 & 0 & f\end{bmatrix}.$$

4. If a, d, f in Problem 3 are all nonzero, show that the only solution to $Ux = 0$ is $x = 0$. Then U has independent columns.

5. Decide the dependence or independence of

(a) the vectors $(1, 3, 2)$, $(2, 1, 3)$, and $(3, 2, 1)$.
(b) the vectors $(1, -3, 2)$, $(2, 1, -3)$, and $(-3, 2, 1)$.

6. Choose three independent columns of U. Then make two other choices. Do the same for A. You have found bases for which spaces?

$$U = \begin{bmatrix}2 & 3 & 4 & 1\\0 & 6 & 7 & 0\\0 & 0 & 0 & 9\\0 & 0 & 0 & 0\end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix}2 & 3 & 4 & 1\\0 & 6 & 7 & 0\\0 & 0 & 0 & 9\\4 & 6 & 8 & 2\end{bmatrix}.$$

7. If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3$, $v_2 = w_1 - w_3$, and $v_3 = w_1 - w_2$ are *dependent*. Find a combination of the v's that gives zero.

8. If w_1, w_2, w_3 are independent vectors, show that the sums $v_1 = w_2 + w_3$, $v_2 = w_1 + w_3$, and $v_3 = w_1 + w_2$ are *independent*. (Write $c_1v_1 + c_2v_2 + c_3v_3 = 0$ in terms of the w's. Find and solve equations for the c's.)

9. Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbf{R}^3$.

(a) These four vectors are dependent because ______.
(b) The two vectors v_1 and v_2 will be dependent if ______.
(c) The vectors v_1 and $(0, 0, 0)$ are dependent because ______.

10. Find two independent vectors on the plane $x + 2y - 3z - t = 0$ in $\mathbf{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

Problems 11–18 are about the space *spanned* by a set of vectors. Take all linear combinations of the vectors.

11. Describe the subspace of $\mathbf{R}^3$ (is it a line or a plane or $\mathbf{R}^3$?) spanned by

(a) the two vectors $(1, 1, -1)$ and $(-1, -1, 1)$.
(b) the three vectors $(0, 1, 1)$ and $(1, 1, 0)$ and $(0, 0, 0)$.
(c) the columns of a 3 by 5 echelon matrix with 2 pivots.
(d) all vectors with positive components.

12. The vector b is in the subspace spanned by the columns of A when there is a solution to ______. The vector c is in the row space of A when there is a solution to ______. *True or false*: If the zero vector is in the row space, the rows are dependent.

13. Find the dimensions of (a) the column space of A, (b) the column space of U, (c) the row space of A, (d) the row space of U. Which two of the spaces are the same?

$$A = \begin{bmatrix}1 & 1 & 0\\1 & 3 & 1\\3 & 1 & -1\end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix}1 & 1 & 0\\0 & 2 & 1\\0 & 0 & 0\end{bmatrix}.$$

14. Choose $x = (x_1, x_2, x_3, x_4)$ in $\mathbf{R}^4$. It has 24 rearrangements like (x_2, x_1, x_3, x_4) and (x_4, x_3, x_1, x_2). Those 24 vectors, including x itself, span a subspace $\mathbf{S}$. Find specific vectors x so that the dimension of $\mathbf{S}$ is: (a) 0, (b) 1, (c) 3, (d) 4.

15. $v + w$ and $v - w$ are combinations of v and w. Write v and w as combinations of $v + w$ and $v - w$. The two pairs of vectors ______ the same space. When are they a basis for the same space?

16. Decide whether or not the following vectors are linearly independent, by solving $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$:

$$v_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \quad v_2 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \quad v_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \quad v_4 = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}.$$

Decide also if they span $\mathbf{R}^4$, by trying to solve $c_1v_1 + \cdots + c_4v_4 = (0, 0, 0, 1)$.

17. Suppose the vectors to be tested for independence are placed into the rows instead of the columns of A. How does the elimination process from A to U decide for or against independence?

18. To decide whether b is in the subspace spanned by $w_1, \dots, w_n$, let the vectors w be the columns of A and try to solve $Ax = b$. What is the result for

(a) $w_1 = (1, 1, 0)$, $w_2 = (2, 2, 1)$, $w_3 = (0, 0, 2)$, $b = (3, 4, 5)$?
(b) $w_1 = (1, 2, 0)$, $w_2 = (2, 5, 0)$, $w_3 = (0, 0, 2)$, $w_4 = (0, 0, 0)$, and any b?

Problems 19–37 are about the requirements for a basis.

19. If $v_1, \dots, v_n$ are linearly independent, the space they span has dimension ______. These vectors are a ______ for that space. If the vectors are the columns of an m by n matrix, then m is ______ than n.

20. Find a basis for each of these subspaces of $\mathbf{R}^4$:

(a) All vectors whose components are equal.
(b) All vectors whose components add to zero.
(c) All vectors that are perpendicular to $(1, 1, 0, 0)$ and $(1, 0, 1, 1)$.
(d) The column space (in $\mathbf{R}^2$) and nullspace (in $\mathbf{R}^5$) of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

21. Find three different bases for the column space of U above. Then find two different bases for the row space of U.

22. Suppose $v_1, v_2, \dots, v_6$ are six vectors in $\mathbf{R}^4$.

(a) Those vectors (do)(do not)(might not) span $\mathbf{R}^4$.
(b) Those vectors (are)(are not)(might be) linearly independent.
(c) Any four of those vectors (are)(are not)(might be) a basis for $\mathbf{R}^4$.
(d) If those vectors are the columns of A, then $Ax = b$ (has) (does not have) (might not have) a solution.

23. The columns of A are n vectors from $\mathbf{R}^m$. If they are linearly independent, what is the rank of A? If they span $\mathbf{R}^m$, what is the rank? If they are a basis for $\mathbf{R}^m$, what then?

24. Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbf{R}^3$. Then find a basis for the intersection of that plane with the xy-plane. Then find a basis for all vectors perpendicular to the plane.

25. Suppose the columns of a 5 by 5 matrix A are a basis for $\mathbf{R}^5$.

(a) The equation $Ax = 0$ has only the solution $x = 0$ because ______.
(b) If b is in $\mathbf{R}^5$ then $Ax = b$ is solvable because ______.

Conclusion: A is invertible. Its rank is 5.

26. Suppose $\mathbf{S}$ is a five-dimensional subspace of $\mathbf{R}^6$. True or false?

(a) Every basis for $\mathbf{S}$ can be extended to a basis for $\mathbf{R}^6$ by adding one more vector.
(b) Every basis for $\mathbf{R}^6$ can be reduced to a basis for $\mathbf{S}$ by removing one vector.

27. U comes from A by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspaces.

28. True or false (give a good reason)?

(a) If the columns of a matrix are dependent, so are the rows.
(b) The column space of a 2 by 2 matrix is the same as its row space.
(c) The column space of a 2 by 2 matrix has the same dimension as its row space.
(d) The columns of a matrix are a basis for the column space.

29. For which numbers c and d do these matrices have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}.$$

30. By locating the pivots, find a basis for the column space of

$$U = \begin{bmatrix} 0 & 5 & 4 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Express each column that is not in the basis as a combination of the basic columns. Find also a matrix A with this echelon form U, but a different column space.

31. Find a counterexample to the following statement: If v_1, v_2, v_3, v_4 is a basis for the vector space $\mathbf{R}^4$, and if $\mathbf{W}$ is a subspace, then some subset of the v's is a basis for $\mathbf{W}$.

32. Find the dimensions of these vector spaces:

(a) The space of all vectors in $\mathbf{R}^4$ whose components add to zero.
(b) The nullspace of the 4 by 4 identity matrix.
(c) The space of all 4 by 4 matrices.

33. Suppose $\mathbf{V}$ is known to have dimension k. Prove that

(a) any k independent vectors in $\mathbf{V}$ form a basis;
(b) any k vectors that span $\mathbf{V}$ form a basis.

In other words, if the number of vectors is known to be correct, either of the two properties of a basis implies the other.

34. Prove that if $\mathbf{V}$ and $\mathbf{W}$ are three-dimensional subspaces of $\mathbf{R}^5$, then $\mathbf{V}$ and $\mathbf{W}$ must have a nonzero vector in common. *Hint*: Start with bases for the two subspaces, making six vectors in all.

35. *True or false*?

(a) If the columns of A are linearly independent, then $Ax = b$ has exactly one solution for every b.
(b) A 5 by 7 matrix never has linearly independent columns.

36. If A is a 64 by 17 matrix of rank 11, how many independent vectors satisfy $Ax = 0$? How many independent vectors satisfy $A^{\mathrm{T}}y = 0$?

37. Find a basis for each of these subspaces of 3 by 3 matrices:

(a) All diagonal matrices.
(b) All symmetric matrices ($A^{\mathrm{T}} = A$).
(c) All skew-symmetric matrices ($A^{\mathrm{T}} = -A$).