Revised 9/23/08

LAB 3: Determinants, Eigenvalues, and Eigenvectors

In this lab you will use MATLAB to study the following topics:

- The determinant function: its properties and how it is calculated from the $PA = LU$ matrix factorization
- The determinant formula for the inverse of a matrix
- Eigenvalues and eigenvectors: the characteristic polynomial of a square matrix, the roots of this polynomial (the eigenvalues), and the eigenvectors.
- Steady-state eigenvector for a regular transition matrix

MATLAB *Preliminaries*

Tcodes: For this lab you will need the Teaching Codes

```
cofactor.m,  splu.m,  nulbasis.m
```

Before opening MATLAB to work on the Lab questions you should copy these codes to your directory. See Lab #2 for further details on how to obtain any of these files from the course home page.

Lab Write-up: After opening your diary file, type the comment line

```
% Math 550 MATLAB Lab Assignment #3
```

at the MATLAB prompt. Type `format compact` so that your diary file will not have unnecessary spaces. Put labels to mark the beginning of your work on each part of each question, so that your edited lab write-up has the format

```
% Question 1 (a) ...
      ⋮
% Question 1 (b) ...
```

and so on. Be sure to answer all the questions in the lab assignment. Insert comments in your diary file as you work through the assignment.

Random Seed: When you start your MATLAB session, initialize the random number generator by typing

```
rand('seed', abcd)
```

where *abcd* are the last four digits of your Student ID number. This will ensure that you generate your own particular random vectors and matrices.

BE SURE TO INCLUDE THIS LINE IN YOUR LAB WRITE-UP

Question 1. The Determinant Function

(a) Row Operations: Generate a 5×5 random integer matrix `A = rmat(5,5)`. Then swap the first and second row of A to get the matrix B using the following commands:

```
B = A; B(2,:)  = A(1,:); B(1,:)  = A(2,:)
```

Use properties of the determinant function to answer the following:

(i) What is the relation between $\det(A)$ and $\det(B)$?

Check your answer by calculating `det(A)` and `det(B)` using MATLAB. Next,let C be the matrix obtained from A by multiplying the first row of A by 10 and adding to the second row of A using the following commands:

```
C = A; C(2,:)  = A(2,:)  + 10*A(1,:)
```

Use properties of the determinant function to answer the following:

(ii) What is the relation between $\det(A)$ and $\det(C)$?

Check your answer by MATLAB. Finally, let D be the matrix obtained from A by multiplying the first row of A by 10:

```
D = A; D(1,:)  = 10*A(1,:)
```

Use properties of the determinant function to answer the following:

(iii) What is the relation between $\det(A)$, $\det(D)$, and $\det(10 * A)$?

Check your answers by MATLAB.

(b) Triangular Matrices: Generate a random 5×5 integer matrix `B = rmat(5,5)`. Then calculate the product

```
B(1,1)*B(2,2)*B(3,3)*B(4,4)*B(5,5)
```

This product is one term in the general formula for $\det(B)$. Can you find $\det(B)$ from this single term? Check by calculating $\det(B)$ with MATLAB. How many arithmetic operations (multiplications and additions) would be needed to calculate $\det(B)$ from formula (6) on page 212 of the text? (In counting these operations, assume that numbers are multiplied and added in pairs, so $3*4*5$ requires two multiplications, and $3+4+5+6$ requires three additions. Multiplying by $\pm$ is free.) *Note:* MATLAB does not use this formula to calculate determinants!

Now form an upper triangular matrix U by

```
U = triu(B)
```

Calculate the product `U(1,1)*U(2,2)*U(3,3)*U(4,4)*U(5,5)`. Can you find $\det(U)$ from this single term? Explain, and then check by calculating $\det(U)$ with MATLAB.

(c) Multiplicative Property: Generate a random 5×5 integer matrix `A = rmat(5,5)`. Then modify A by setting `A(1,1)=0; A(2,1) = 0`. The reduction of the (modified) matrix A to row echelon form can be expressed in terms of a matrix factorization as $PA = LU$. Here P is a *permutation matrix* that expresses the row interchanges that are needed to apply Gaussian elimination to A, and L and U give the LU decomposition of PA.

(i) For the modified matrix A, explain why P will not be the identity matrix.

You can calculate the $PA = LU$ factorization by using the T-code `splu.m`:

```
[P, L, U, sign] = splu(A)
```

Here `sign` gives $\det(P)$, which is $+1$ for an even number of row interchanges to transform P into the identity matrix, and -1 for an odd number of row interchanges. Check (by MATLAB) that $PA = LU$. Then write comments to answer the following.

(ii) What is $\det(P)$? Why? Compare your answer with the value of `sign` that MATLAB has calculated.

(iii) What is $\det(L)$? Why?

(iv) What is the relation between $\det(A)$ and $\det(U)$? Why?

Check your answer to *(iv)* by MATLAB.

Question 2. Cofactor Matrix and Cramer's Rule

(a) Cofactor Expansion for Determinants: The Teaching Code m-file `cofactor.m` calculates the matrix of cofactors of a square matrix. Generate a random 4×4 integer matrix `a = rmat(4,4)`. Then use MATLAB to calculate the cofactor matrix `A = cofactor(a)`. Use MATLAB to calculate the two sums

a(1,1)*A(1,1) + a(1,2)*A(1,2) + a(1,3)*A(1,3) + a(1,4)*A(1,4)
a(2,1)*A(2,1) + a(2,2)*A(2,2) + a(2,3)*A(2,3) + a(2,4)*A(2,4)

(You can do this efficiently using the colon operator to work with entire rows of a and A).

(i) Explain why both sums give the same value.

Now use MATLAB to calculate the sum

a(2,1)*A(1,1) + a(2,2)*A(1,2) + a(2,3)*A(1,3) + a(2,4)*A(1,4)

(the inner product of the *second* row of a with the *first* row of A). Explain why the answer is zero by creating a matrix b with $\det(b) = 0$ such that the sum just calculated is the cofactor expansion along the first row of $\det(b)$. Use MATLAB to calculate `cofactor(b)` and show that the cofactors of the first rows of a and b are the same.

(b) Adjugate Matrix: The transposed cofactor matrix A' is called the *adjugate matrix* $\mathrm{adj}(a)$. Use MATLAB to calculate $a * A'$. Use formula (2) on page 221 of Strang to explain the answer you get. Explain how this matrix product result includes the calculations in part **(a)** as special cases.

(c) Inverse Matrix and Cramer's Rule:

Set `M = A'/det(a)`. From the result of part **(b)**, what is the matrix `a*M`? Check by MATLAB.

Generate a 3×3 random integer matrix `B = rmat(3,3)` and a random column vector **c** `= rvect(3)` Define matrices $BC1, BC2, BC3$ by replacing the first (respectively second or third) column of B by the vector **c**:

```
BC1 = B; BC2 = B; BC3 = B;
BC1(:,1) = c, BC2(:,2) = c, BC3(:,3) = c
```

Define the column vector

```
x = [det(BC1)/det(B); det(BC2)/det(B); det(BC3)/det(B)]
```

and calculate $B * \mathbf{x}$. Explain the result using *Cramer's Rule* (see Strang, page 221).

Question 3. Eigenvalues and Eigenvectors

(a) Graphic Demo: Type `eigshow` at the MATLAB prompt. A graphics window should open. Underneath the graph the statement

```
Make A*x parallel to x
```

should appear (if it does not, then click on the `eig` button to get this statement).

Click on the pull-down bar above the graph and select the matrix `[1 3; 4 2]/4`. Move the cursor onto the vector $\mathbf{x}$, and make $\mathbf{x}$ go around a full circle. The transformed vector $A\mathbf{x}$ then moves around an ellipse. Search for the *special directions* where $A\mathbf{x}$ and $\mathbf{x}$ lie on a straight line. When $\mathbf{x}$ points in one of these directions, it is an *eigenvector* of the matrix A (the word *eigen* means *special* in German). For $\mathbf{x}$ pointing in these *special directions*, $A\mathbf{x} = \lambda\mathbf{x}$, where λ is an *eigenvalue* of A. Since $\mathbf{x}$ is a unit vector, the length of $A\mathbf{x}$ is $|\lambda|$. If $A\mathbf{x}$ points in the same direction as $\mathbf{x}$, then $\lambda > 0$. If $A\mathbf{x}$ points in the opposite direction to $\mathbf{x}$, then $\lambda < 0$.

From your graphical experimentation answer the following questions (no algebraic calculations needed):

(i) How many positive eigenvalues does A have?

(ii) How many negative eigenvalues does A have?

(iii) What are the (approximate) numerical values of the eigenvalues?

(Don't try to print the eigshow window.)

(b) Characteristic Polynomial: Generate a random 2×2 integer matrix `A = rmat(2,2)`. The eigenvalues of A are the roots of the *characteristic polynomial* of A. Calculate the characteristic polynomial $p(s) = \det(A - sI_2)$ by hand. Verify that the constant term in the polynomial $p(s)$ is $\det(A)$. Verify that the coefficient of s is $-\mathrm{trace}(A)$ (the *trace* is the sum of the diagonal entries). *Prove* (by hand calculation) that the roots of $p(s)$ will be *real* if and only if

$$(\mathrm{trace}(A))^2 \geq 4 * \det(A).$$

Is this condition satisfied for your matrix A?

(c) Companion Matrices: Every polynomial of degree n with leading coefficient $(-1)^n$ is the characteristic polynomial of a suitable matrix. For example, the polynomial

$$p(s) = s^4 - 2s^3 - 13s^2 + 14s + 24$$

which is represented in MATLAB as a row vector of the coefficients of p ordered by descending powers of the variable:

```
p = [1   -2   -13   14   24 ]
```

is associated with the 4×4 *companion matrix* generated by the MATLAB command `A = compan(p)`.

Show by hand calculation that $\det(A - sI_4) = p(s)$ (use the cofactor expansion of the determinant along the first row). Thus $p(s)$ is the *characteristic polynomial* of A. Now use the MATLAB command `poly(A)` to generate the characteristic polynomial of A and verify that you get the same result.

(d) Eigenvalues and Eigenvectors: Let A be the companion matrix from part **(c)**. Use the MATLAB command `[S  D] = eig(A)` to generate a matrix S whose columns are the eigenvectors of A, and a diagonal matrix D whose diagonal entries are the eigenvalues of A. Verify by hand calculation that the characteristic polynomial factors as

$$p(s) = (\lambda_1 - s)(\lambda_2 - s)(\lambda_3 - s)(\lambda_4 - s),$$

where λ_i $(i = 1, 2, 3, 4)$ are the eigenvalues. Verify that $A = S * D * S^{-1}$ (see Strang page 245, 5C). How can you express A^{10} in terms of D and S? Explain and verify your answer using MATLAB.

Question 4. Steady-State Eigenvector for a Regular Transition Matrix

(a) A matrix is called a *transition matrix* if its entries are nonnegative and the sum of the entries in each column is one. A transition matrix is called *regular* if all its entries are strictly positive. For this question generate a random 2×2 transition matrix A by

```
A = eye(2); B = rand(2);
A(:,1) = B(:,1)/sum(B(:,1)); A(:,2) = B(:,2)/sum(B(:,2))
```

Calculate `[1  1]*A` . Show by a hand calculation why the answer proves that A is a transition matrix. Since all the entries in A are positive, it is a *regular* transition matrix.

(b) A regular transition matrix always has 1 as its largest eigenvalue. Use the T-code `nulbasis` to calculate a normalized eigenvector for the matrix A you generated in part **(a)**.

```
u = nulbasis(A - eye(2)), v = u/sum(u)
```

The vector $\mathbf{v}$ should have components that are positive and sum to 1. Verify by MATLAB that $A\mathbf{v} = \mathbf{v}$. Thus $\mathbf{v}$ is an eigenvector for A with eigenvalue 1, called the *steady-state vector* for A. Plot this vector (as a solid line) by

```
plot([0,v(1)], [0, v(2)]), hold on
```

Leave the graphic window open for the next part.

(c) A general result about regular transition matrices asserts that if **p** is any initial choice of a probability vector in $\mathbf{R}^2$, then the sequence of vectors $A^k\mathbf{p}$ converges to the steady-state vector **v** as $k \to \infty$. To demonstrate this graphically for your matrix A, generate a random initial probability vector

```
w = rand(2,1), p = w/sum(w)
```

Now graph the vector $A\mathbf{p}$ (as a dotted line) in the same window from part **(b)**:

```
p = A*p, plot([0,p(1)], [0, p(2)], ':'), hold on
```

To plot the sequence of vectors $A^2\mathbf{p}$, $A^3\mathbf{p}, A^4\mathbf{p}, \ldots$ in the graphics window, just use the up-arrow key $\uparrow$ to repeat this last command. Do this as many times as needed until the vector **p** has converged numerically (to three decimal places in each component) to the steady-state vector **v** that you plotted in part **(b)**. Print the graphics window and include it in your lab report.

Final Editing of Lab Write-up:

After you have worked through all the parts of the lab assignment, edit your diary file. Include the Matlab calculations, but remove errors that you made in entering commands and remove other material that is not directly related to the questions.