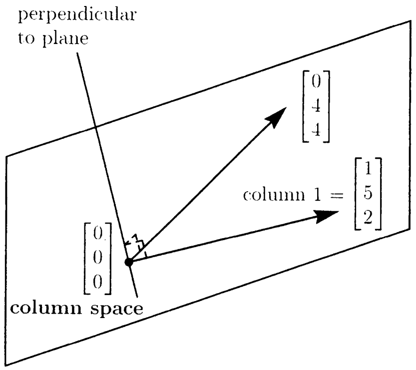

Figure 2.1 The column space $C(A)$, a plane in three-dimensional space.

$C(A)$. Requirements (i) and (ii) for a subspace of $\mathbf{R}^m$ are easy to check:

(i) Suppose b and b' lie in the column space, so that $Ax = b$ for some x and $Ax' = b'$ for some x'. Then $A(x + x') = b + b'$, so that $b + b'$ is also a combination of the columns. The column space of all attainable vectors b is closed under addition.

(ii) If b is in the column space $C(A)$, so is any multiple cb. If some combination of columns produces b (say $Ax = b$), then multiplying that combination by c will produce cb. In other words, $A(cx) = cb$.

For another matrix A, the dimensions in Figure 2.1 may be very different. The smallest possible column space (one vector only) comes from the zero matrix $A = 0$. The only combination of the columns is $b = 0$. At the other extreme, suppose A is the 5 by 5 identity matrix. Then $C(I)$ is the whole of $\mathbf{R}^5$; the five columns of I can combine to produce any five-dimensional vector b. This is not at all special to the identity matrix. *Any 5 by 5 matrix that is nonsingular will have the whole of* $\mathbf{R}^5$ *as its column space.* For such a matrix we can solve $Ax = b$ by Gaussian elimination; there are five pivots. Therefore every b is in $C(A)$ for a nonsingular matrix.

You can see how Chapter 1 is contained in this chapter. There we studied n by n matrices whose column space is $\mathbf{R}^n$. Now we allow singular matrices, and rectangular matrices of any shape. Then $C(A)$ can be somewhere between the zero space and the whole space $\mathbf{R}^m$. Together with its perpendicular space, it gives one of our two approaches to understanding $Ax = b$.

The Nullspace of A

The second approach to $Ax = b$ is "dual" to the first. We are concerned not only with attainable right-hand sides b, but also with the solutions x that attain them. The right-hand side $b = 0$ always allows the solution $x = 0$, but there may be infinitely many other solutions. (There always are, if there are more unknowns than equations, $n > m$.) ***The solutions to $Ax = 0$ form a vector space—the nullspace of A.***

The ***nullspace*** of a matrix consists of all vectors x such that $Ax = 0$. It is denoted by $N(A)$. It is a subspace of $\mathbf{R}^n$, just as the column space was a subspace of $\mathbf{R}^m$.

Requirement (i) holds: If $Ax = 0$ and $Ax' = 0$, then $A(x + x') = 0$. Requirement (ii) also holds: If $Ax = 0$ then $A(cx) = 0$. Both requirements fail if the right-hand side is not zero! Only the solutions to a *homogeneous* equation ($b = 0$) form a subspace. The nullspace is easy to find for the example given above; it is as small as possible:

$$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The first equation gives $u = 0$, and the second equation then forces $v = 0$. The nullspace contains only the vector $(0, 0)$. This matrix has "independent columns"—a key idea that comes soon.

The situation is changed when a third column is a combination of the first two:

$$\textbf{Larger nullspace} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}.$$

B has the same column space as A. The new column lies in the plane of Figure 2.1; it is the sum of the two column vectors we started with. But the nullspace of B contains the vector $(1, 1, -1)$ and automatically contains any multiple $(c, c, -c)$:

$$\textbf{Nullspace is a line} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} c \\ c \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The nullspace of B is the line of all points $x = c$, $y = c$, $z = -c$. (The line goes through the origin, as any subspace must.) We want to be able, for any system $Ax = b$, to find $C(A)$ and $N(A)$: all attainable right-hand sides b and all solutions to $Ax = 0$.

The vectors b are in the column space and the vectors x are in the nullspace. We shall compute the dimensions of those subspaces and a convenient set of vectors to generate them. We hope to end up by understanding all *four* of the subspaces that are intimately related to each other and to A—the column space of A, the nullspace of A, and their two perpendicular spaces.

Problem Set 2.1

1. Construct a subset of the x-y plane $\mathbf{R}^2$ that is

(a) closed under vector addition and subtraction, but not scalar multiplication.

(b) closed under scalar multiplication but not under vector addition.

Hint: Starting with u and v, add and subtract for (a). Try cu and cv for (b).

2. Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0$.

(b) The plane of vectors b with $b_1 = 1$.
(c) The vectors b with $b_2 b_3 = 0$ (this is the union of two subspaces, the plane $b_2 = 0$ and the plane $b_3 = 0$).
(d) All combinations of two given vectors $(1, 1, 0)$ and $(2, 0, 1)$.
(e) The plane of vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0$.

3. Describe the column space and the nullspace of the matrices

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

4. What is the smallest subspace of 3 by 3 matrices that contains all symmetric matrices *and* all lower triangular matrices? What is the largest subspace that is contained in both of those subspaces?

5. Addition and scalar multiplication are required to satisfy these eight rules:

1. $x + y = y + x$.
2. $x + (y + z) = (x + y) + z$.
3. There is a unique "zero vector" such that $x + 0 = x$ for all x.
4. For each x there is a unique vector $-x$ such that $x + (-x) = 0$.
5. $1x = x$.
6. $(c_1 c_2)x = c_1(c_2 x)$.
7. $c(x + y) = cx + cy$.
8. $(c_1 + c_2)x = c_1 x + c_2 x$.

(a) Suppose addition in $\mathbf{R}^2$ adds an extra 1 to each component, so that $(3, 1) + (5, 0)$ equals $(9, 2)$ instead of $(8, 1)$. With scalar multiplication unchanged, which rules are broken?
(b) Show that the set of all positive real numbers, with $x + y$ and cx redefined to equal the usual xy and x^c, is a vector space. What is the "zero vector"?
(c) Suppose $(x_1, x_2) + (y_1, y_2)$ is defined to be $(x_1 + y_2, x_2 + y_1)$. With the usual $cx = (cx_1, cx_2)$, which of the eight conditions are not satisfied?

6. Let $\mathbf{P}$ be the plane in 3-space with equation $x + 2y + z = 6$. What is the equation of the plane $\mathbf{P}_0$ through the origin parallel to $\mathbf{P}$? Are $\mathbf{P}$ and $\mathbf{P}_0$ subspaces of $\mathbf{R}^3$?

7. Which of the following are subspaces of $\mathbf{R}^\infty$?
(a) All sequences like $(1, 0, 1, 0, \ldots)$ that include infinitely many zeros.
(b) All sequences $(x_1, x_2, \ldots)$ with $x_j = 0$ from some point onward.
(c) All decreasing sequences: $x_{j+1} \le x_j$ for each j.
(d) All convergent sequences: the x_j have a limit as $j \to \infty$.
(e) All arithmetic progressions: $x_{j+1} - x_j$ is the same for all j.
(f) All geometric progressions $(x_1, kx_1, k^2 x_1, \ldots)$ allowing all k and x_1.

8. Which of the following descriptions are correct? The solutions x of

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

form
(a) a plane.
(b) a line.
(c) a point.
(d) a subspace.
(e) the nullspace of A.
(f) the column space of A.

9. Show that the set of nonsingular 2 by 2 matrices is not a vector space. Show also that the set of *singular* 2 by 2 matrices is not a vector space.

10. The matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is a "vector" in the space $\mathbf{M}$ of all 2 by 2 matrices. Write the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$. What matrices are in the smallest subspace containing A?

11. (a) Describe a subspace of $\mathbf{M}$ that contains $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but not $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.
(b) If a subspace of $\mathbf{M}$ contains A and B, must it contain I?
(c) Describe a subspace of $\mathbf{M}$ that contains no nonzero diagonal matrices.

12. The functions $f(x) = x^2$ and $g(x) = 5x$ are "vectors" in the vector space $\mathbf{F}$ of all real functions. The combination $3f(x) - 4g(x)$ is the function $h(x) =$ ______. Which rule is broken if multiplying $f(x)$ by c gives the function $f(cx)$?

13. If the sum of the "vectors" $f(x)$ and $g(x)$ in $\mathbf{F}$ is defined to be $f(g(x))$, then the "zero vector" is $g(x) = x$. Keep the usual scalar multiplication $cf(x)$, and find two rules that are broken.

14. Describe the smallest subspace of the 2 by 2 matrix space $\mathbf{M}$ that contains
(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
(c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.
(d) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

15. Let $\mathbf{P}$ be the plane in $\mathbf{R}^3$ with equation $x + y - 2z = 4$. The origin $(0, 0, 0)$ is not in $\mathbf{P}$! Find two vectors in $\mathbf{P}$ and check that their sum is not in $\mathbf{P}$.

16. $\mathbf{P}_0$ is the plane through $(0, 0, 0)$ parallel to the plane $\mathbf{P}$ in Problem 15. What is the equation for $\mathbf{P}_0$? Find two vectors in $\mathbf{P}_0$ and check that their sum is in $\mathbf{P}_0$.

17. The four types of subspaces of $\mathbf{R}^3$ are planes, lines, $\mathbf{R}^3$ itself, or $\mathbf{Z}$ containing only $(0, 0, 0)$.
(a) Describe the three types of subspaces of $\mathbf{R}^2$.
(b) Describe the five types of subspaces of $\mathbf{R}^4$.

18. (a) The intersection of two planes through $(0, 0, 0)$ is probably a ______ but it could be a ______. It can't be the zero vector $\mathbf{Z}$!
(b) The intersection of a plane through $(0, 0, 0)$ with a line through $(0, 0, 0)$ is probably a ______ but it could be a ______.
(c) If $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{R}^5$, their intersection $\mathbf{S} \cap \mathbf{T}$ (vectors in both subspaces) is a subspace of $\mathbf{R}^5$. *Check the requirements on $x + y$ and cx.*

19. Suppose **P** is a plane through (0, 0, 0) and **L** is a line through (0, 0, 0). The smallest vector space containing both **P** and **L** is either _____ or _____.

20. True or false for **M** = all 3 by 3 matrices (check addition using an example)?

(a) The skew-symmetric matrices in **M** (with $A^T = -A$) form a subspace.
(b) The unsymmetric matrices in **M** (with $A^T \neq A$) form a subspace.
(c) The matrices that have (1, 1, 1) in their nullspace form a subspace.

Problems 21–30 are about column spaces $C(A)$ and the equation $Ax = b$.

21. Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

22. For which right-hand sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$\text{(a)} \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \qquad \text{(b)} \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

23. Adding row 1 of A to row 2 produces B. Adding column 1 to column 2 produces C. A combination of the columns of _____ is also a combination of the columns of A. Which two matrices have the same column _____?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

24. For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

25. (Recommended) If we add an extra column b to a matrix A, then the column space gets larger unless _____. Give an example in which the column space gets larger and an example in which it doesn't. Why is $Ax = b$ solvable exactly when the column space *doesn't* get larger by including b?

26. The columns of AB are combinations of the columns of A. This means: *The column space of AB is contained in* (possibly equal to) *the column space of A*. Give an example where the column spaces of A and AB are not equal.

27. If A is any 8 by 8 invertible matrix, then its column space is _____. Why?

28. True or false (with a counterexample if false)?

(a) The vectors b that are not in the column space $C(A)$ form a subspace.
(b) If $C(A)$ contains only the zero vector, then A is the zero matrix.
(c) The column space of $2A$ equals the column space of A.
(d) The column space of $A - I$ equals the column space of A.

29. Construct a 3 by 3 matrix whose column space contains (1, 1, 0) and (1, 0, 1) but not (1, 1, 1). Construct a 3 by 3 matrix whose column space is only a line.

30. If the 9 by 12 system $Ax = b$ is solvable for every b, then $C(A) =$ _____.

31. Why isn't $\mathbf{R}^2$ a subspace of $\mathbf{R}^3$?

2.2 SOLVING $Ax = 0$ AND $Ax = b$

Chapter 1 concentrated on square invertible matrices. There was one solution to $Ax = b$, and it was $x = A^{-1}b$. That solution was found by elimination (not by computing A^{-1}). A rectangular matrix brings new possibilities—U may not have a full set of pivots. This section goes onward from U to a reduced form R—**the simplest matrix that elimination can give**. R reveals all solutions immediately.

For an invertible matrix, the nullspace contains only $x = 0$ (multiply $Ax = 0$ by A^{-1}). The column space is the whole space ($Ax = b$ has a solution for every b). The new questions appear when the nullspace contains *more than the zero vector* and/or the column space contains *less than all vectors*:

1. Any vector x_n in the nullspace can be added to a particular solution x_p. The solutions to all linear equations have this form, $x = x_p + x_n$:

Complete solution $\quad Ax_p = b$ **and** $Ax_n = 0$ **produce** $A(x_p + x_n) = b$.

2. When the column space doesn't contain every b in $\mathbf{R}^m$, we need the conditions on b that make $Ax = b$ solvable.

A 3 by 4 example will be a good size. We will write down all solutions to $Ax = 0$. We will find the conditions for b to lie in the column space (so that $Ax = b$ is solvable). The 1 by 1 system $0x = b$, one equation and one unknown, shows two possibilities:

$0x = b$ has *no solution* unless $b = 0$. The column space of the 1 by 1 zero matrix contains only $b = 0$.

$0x = 0$ has *infinitely many solutions*. The nullspace contains *all* x. A particular solution is $x_p = 0$, and the complete solution is $x = x_p + x_n = 0 +$ (any x).

Simple, I admit. If you move up to 2 by 2, it's more interesting. The matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not invertible: $y + z = b_1$ and $2y + 2z = b_2$ usually have no solution.

There is ***no solution*** unless $b_2 = 2b_1$. The column space of A contains only those b's, the multiples of (1, 2).

When $b_2 = 2b_1$ there are ***infinitely many solutions***. A particular solution to $y + z = 2$ and $2y + 2z = 4$ is $x_p = (1, 1)$. The nullspace of A in Figure 2.2 contains $(-1, 1)$ and all its multiples $x_n = (-c, c)$:

Complete solution $\quad \begin{matrix} y + z = 2 \\ 2y + 2z = 4 \end{matrix}$ is solved by $\quad x_p + x_n = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - c \\ 1 + c \end{bmatrix}.$

Another Worked Example

The full picture uses elimination and pivot columns to find the column space, nullspace, and rank. The 3 by 4 matrix A has rank 2:

$$Ax = b \quad \text{is} \quad \begin{aligned} \mathbf{1}x_1 + \mathbf{2}x_2 + \mathbf{3}x_3 + \mathbf{5}x_4 &= b_1 \\ \mathbf{2}x_1 + \mathbf{4}x_2 + \mathbf{8}x_3 + \mathbf{12}x_4 &= b_2 \\ \mathbf{3}x_1 + \mathbf{6}x_2 + \mathbf{7}x_3 + \mathbf{13}x_4 &= b_3 \end{aligned} \qquad (6)$$

1. Reduce $[A\ \ b]$ to $[U\ \ c]$, to reach a triangular system $Ux = c$.
2. Find the condition on b_1, b_2, b_3 to have a solution.
3. Describe the column space of A: Which plane in $\mathbf{R}^3$?
4. Describe the nullspace of A: Which special solutions in $\mathbf{R}^4$?
5. Find a particular solution to $Ax = (0, 6, -6)$ and the complete $x_p + x_n$.
6. Reduce $[U\ \ c]$ to $[R\ \ d]$: Special solutions from R and x_p from d.

Solution (Notice how the right-hand side is included as an extra column!)

1. The multipliers in elimination are 2 and 3 and -1, taking $[A\ \ b]$ to $[U\ \ c]$.

$$\begin{bmatrix} 1 & 2 & 3 & 5 & \boldsymbol{b_1} \\ 2 & 4 & 8 & 12 & \boldsymbol{b_2} \\ 3 & 6 & 7 & 13 & \boldsymbol{b_3} \end{bmatrix} \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & \boldsymbol{b_1} \\ 0 & 0 & 2 & 2 & \boldsymbol{b_2 - 2b_1} \\ 0 & 0 & -2 & -2 & \boldsymbol{b_3 - 3b_1} \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & \boldsymbol{b_1} \\ 0 & 0 & 2 & 2 & \boldsymbol{b_2 - 2b_1} \\ 0 & 0 & 0 & 0 & \boldsymbol{b_3 + b_2 - 5b_1} \end{array}\right].$$

2. The last equation shows the solvability condition $b_3 + b_2 - 5b_1 = 0$. Then $0 = 0$.
3. The column space of A is the plane containing all combinations of the pivot columns $(1, 2, 3)$ and $(3, 8, 7)$. **Second description**: The column space contains all vectors with $b_3 + b_2 - 5b_1 = 0$. That makes $Ax = b$ solvable, so b is in the column space. *All columns of A pass this test $b_3 + b_2 - 5b_1 = 0$. This is the equation for the plane (in the first description of the column space).*
4. The special solutions in N have free variables $x_2 = 1, x_4 = 0$ and $x_2 = 0, x_4 = 1$:

Nullspace matrix
Special solutions to $Ax = 0$
Back-substitution in $Ux = 0$
Just switch signs in $Rx = 0$

$$N = \begin{bmatrix} -2 & -2 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

5. Choose $b = (0, 6, -6)$, which has $b_3 + b_2 - 5b_1 = 0$. Elimination takes $Ax = b$ to $Ux = c = (0, 6, 0)$. Back-substitute with free variables $= 0$:

Particular solution to $Ax_p = (0, 6, -6)$

$$x_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} \begin{matrix} \\ \text{free} \\ \\ \text{free} \end{matrix}$$

The complete solution to $Ax = (0, 6, -6)$ is (this x_p) + (all x_n).

6. In the reduced R, the third column changes from $(3, 2, 0)$ to $(0, 1, 0)$. The right-hand side $c = (0, 6, 0)$ becomes $d = (-9, 3, 0)$. Then -9 and 3 go into x_p:

$$[U\ \ c] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow [R\ \ d] = \left[\begin{array}{cccc|c} \mathbf{1} & 2 & 0 & 2 & \mathbf{-9} \\ 0 & 0 & \mathbf{1} & 1 & \mathbf{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

That final matrix $[R\ \ d]$ is rref($[A\ \ b]$) = rref($[U\ \ c]$). The numbers 2 and 0 and 2 and 1 in the free columns of R have opposite sign in the special solutions (the nullspace matrix N). Everything is revealed by $Rx = d$.

Problem Set 2.2

1. Construct a system with more unknowns than equations, but no solution. Change the right-hand side to zero and find all solutions x_n.

2. Reduce A and B to echelon form, to find their ranks. Which variables are free?

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Find the special solutions to $Ax = 0$ and $Bx = 0$. Find all solutions.

3. Find the echelon form U, the free variables, and the special solutions:

$$A = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

$Ax = b$ is consistent (has a solution) when b satisfies $b_2 =$ ______. Find the complete solution in the same form as equation (4).

4. Carry out the same steps as in the previous problem to find the complete solution of $Mx = b$:

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

5. Write the complete solutions $x = x_p + x_n$ to these systems, as in equation (4):

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

6. Describe the set of attainable right-hand sides b (in the column space) for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

by finding the constraints on b that turn the third equation into $0 = 0$ (after elimination). What is the rank, and a particular solution?

7. Find the value of c that makes it possible to solve $Ax = b$, and solve it:

$$\begin{aligned} u + v + 2w &= 2 \\ 2u + 3v - w &= 5 \\ 3u + 4v + w &= c. \end{aligned}$$