Math 642:550 — Summer 2001
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A robust method for finding characteristic polynomials

Introduction. Notes on this topic prepared for this course in Summer 1999 and revised in Summer 2000, Both versions are available from the course web page. Here, we introduce some minor modifications and new exercises. The main point, already present in the earlier notes, is that the definition of *characteristic polynomial* is ill-suited to computation and to describe a calculation the leads to reliable answers with hand computation and is easily programmed for machine calculation.

More details on the method can be found in F. R. Gantmacher, *The Theory of Matrices*, Chelsea, 1990, § IV.5. The method is attributed to Leverrier, though most expositions use a more efficient form due to D. K. Faddeev.

In the 1999 notes, Leverrier's version of the calculation was worked out in painful detail. However, this approach serves mainly to set the stage for Faddeev's version, so only a brief summary will be given. (In the notes from Summer 2000, only Faddeev's version was given.)

This approach is so well suited to human calculation that it was rediscovered often. The discussion in Alton S. Householder, *The Theory of Matrices in Numerical Analysis*, Dover, New York, 1975. (ISBN 0-486-61781-5) has the best list of published rediscoveries that I have seen.

Powers and traces. We begin by collecting some facts that are easily verified in the case when the characteristic polynomial has no repeated roots. Extending the proofs to apply to all square matrices is a bother, but this may be ignored in practice, since the method always verifies that the characteristic polynomial it finds is correct. A proof would only serve to assure us that the method cannot fail if the computations are done correctly.

If you really need a proof, it is possible to obtain formulas for the matrices arising in the calculation and use these formulas to prove the validity of the method.

The key to the method is that the sum of the elements on the diagonal of a square matrix M, called the **trace** of M is also equal to the sum of the eigenvalues of M. By itself, this isn't too useful. However, we also know that the eigenvalues of M^k are the k-th powers of the eigenvalues of M. Thus computing the traces of powers of M gives the sums of powers of the roots of the characteristic polynomial.

A small example will be useful. Let

$$M = \begin{bmatrix} \mathbf{3} & 1 & 5 \\ 3 & \mathbf{3} & 1 \\ 4 & 6 & \mathbf{4} \end{bmatrix}.$$

Then,

$$M^2 = \begin{bmatrix} \mathbf{32} & 36 & 36 \\ 22 & \mathbf{18} & 22 \\ 46 & 46 & \mathbf{42} \end{bmatrix} \qquad M^3 = \begin{bmatrix} \mathbf{348} & 356 & 340 \\ 208 & \mathbf{208} & 216 \\ 444 & 436 & \mathbf{444} \end{bmatrix}.$$

If the eigenvalues of M are denoted $\lambda_1, \lambda_2, \lambda_3$, then equating traces and power sums gives

$$\lambda_1 + \lambda_2 + \lambda_3 = 10$$
$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 92$$
$$\lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 1000$$

The coefficients of the characteristic polynomial that we want to determine are (up to sign) the *elementary symmetric functions* $\sigma_1 = \lambda_1 + \lambda_2 + \lambda_3$, $\sigma_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$, and $\sigma_3 = \lambda_1\lambda_2\lambda_3$. Immediately, we have $\sigma_1 = 10$, and without trying too hard, one finds

$$2\sigma_2 = (\lambda_1 + \lambda_2 + \lambda_3)^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) = 10^2 - 92 = 8.$$

Thus, $\sigma_2 = 4$. Generating some expressions of degree 3, and introducing some obvious abbreviations,

$$\Big(\sum \lambda_i\Big)\Big(\sum \lambda_i^2\Big) = \sum \lambda_i^3 + \sum \lambda_i^2\lambda_j$$
$$\Big(\sum \lambda_i\Big)\Big(\sum \lambda_i\lambda_j\Big) = \sum \lambda_i^2\lambda_j + 3\sigma_3$$

Thus, $\sum \lambda_i^2\lambda_j = (10)(92) - 1000 = -80$ and $3\sigma_3 = (10)(4) - (-80) = 120$, so $\sigma_3 = 40$. This gives the characteristic polynomial to be $\lambda^3 - 10\lambda^2 + 4\lambda - 40 = (\lambda - 10)(\lambda^2 + 4)$ — if the calculations are correct.

The Newton identities. The relations between symmetric polynomials used in our example were generalized and put in a systematic form by Newton. It is more convenient to change notation to emphasize the coefficients of the polynomial rather than the symmetric functions. To be consistent with the reference, we write the characteristic polynomial in the form

$$\lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - \cdots - p_n,$$

with negative signs in front of all terms except the leading term, which has coefficient $+1$. In particular, the determinant turns out to be $(-1)^{n-1}p_n$. We also introduce s_k to stand for the sum of the k-th powers of the roots. The first few formulas are

$$p_1 = s_1$$
$$2p_2 = s_2 - p_1 s_1$$
$$3p_3 = s_3 - p_1 s_2 - p_2 s_1$$

and, for each k,

$$kp_k = s_k - \sum \{\, p_i s_j : i > 0,\ j > 0, i + j = k \,\}$$

If these identities are used in order, then everything needed on the right side of an identity will be either a given s_n or a p_k obtained at a previous step. We will accept these identities on good authority and not prove them at this time.

Faddeev's improvement. Instead of applying Newton's identities after computing all the s_k, we construct the p_k and a a sequence of matrices $F_k(M)$ such that

$$F_1(M) = M \qquad (B)$$
$$p_k = \frac{1}{k}\,\mathrm{tr}\,\big(F_k(M)\big) \qquad (T_k)$$
$$F_{k+1}(M) = M(F_k(M) - p_k I) \qquad (M_k)$$

Here, (B) is the basis of the inductive construction. It tells us what F_1 is. Then for $k = 1, 2, 3, \ldots$ we apply first (T_k) to use F_k to identify p_k and then (M_k) to determine F_{k+1}. The only part of the algorithm that changes with k is the division by k in (T_k). This inductive definition is used to assure that

$$F_k(M) = M^k - p_1 M^{k-1} - p_2 M^{k-2} - \cdots - p_{k-1}M.$$

The Newton identities then show that (T_k) gives the correct value of p_k.

One nice feature of this algorithm is that it is self-checking. The Cayley-Hamilton theorem (mentioned in our text only in exercises — see the index) says that a matrix satisfies its characteristic polynomial. In the context of this method, this says that F_{n+1} must turn out to be identically zero. If you fail to obtain a zero matrix at this stage, then you know that you have made a mistake in your calculation. However, the operations in the algorithm consist only of modifying the diagonal of a matrix and multiplying by a fixed matrix.

Revisiting the first example. Here is how the previous example looks when Faddeev's method is used.

$$F_1(M) = \begin{bmatrix} \mathbf{3} & 1 & 5 \\ 3 & \mathbf{3} & 1 \\ 4 & 6 & \mathbf{4} \end{bmatrix} \tag{B}$$

$$p_1 = (3+3+4)/1 = 10 \tag{T_1}$$

$$F_2(M) = \begin{bmatrix} \mathbf{3} & 1 & 5 \\ 3 & \mathbf{3} & 1 \\ 4 & 6 & \mathbf{4} \end{bmatrix} \begin{bmatrix} \mathbf{-7} & 1 & 5 \\ 3 & \mathbf{-7} & 1 \\ 4 & 6 & \mathbf{-6} \end{bmatrix} = \begin{bmatrix} \mathbf{2} & 26 & -14 \\ -8 & \mathbf{-12} & 12 \\ 6 & -14 & \mathbf{2} \end{bmatrix} \tag{M_1}$$

$$p_2 = (2-12+2)/2 = -4 \tag{T_2}$$

$$F_3(M) = \begin{bmatrix} \mathbf{3} & 1 & 5 \\ 3 & \mathbf{3} & 1 \\ 4 & 6 & \mathbf{4} \end{bmatrix} \begin{bmatrix} \mathbf{6} & 26 & -14 \\ -8 & \mathbf{-8} & 12 \\ 6 & -14 & \mathbf{6} \end{bmatrix} = \begin{bmatrix} \mathbf{40} & 0 & 0 \\ 0 & \mathbf{40} & 0 \\ 0 & 0 & \mathbf{40} \end{bmatrix} \tag{M_2}$$

$$p_3 = (40+40+40)/3 = 40 \tag{T_3}$$

and $F_4(M) = 0$, as expected.

Exercises. In order to check that you have organized the steps of Faddeev's algorithm correctly, begin with some examples where you know the characteristic polynomial

B1 A 3 by 3 identity matrix.

B2 The 4 by 4 matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then try a more general 4 by 4 matrix.

B3 The matrix

$$\begin{bmatrix} 3 & 1 & 5 & -2 \\ 3 & 3 & 0 & 1 \\ 4 & 6 & -4 & 3 \\ 2 & -1 & -2 & 0 \end{bmatrix}$$

Sketch of a proof. This section may be skipped. It is included only to show how the gaps in the theory may be filled in.

When you know exactly what a calculation is supposed to do, you have good chance of proving that has all the required properties. As a result of some experiments, a conjecture was formulated about the form of the entries of all $F_k(M)$. It looks messy at first, but when read in the context of the proof, it is easier to see how it emerged from exploratory computations.

The k by k matrix $F_k(M)$ will be expressed as a sum of matrices F_U with at most one nonzero entry, with many such matrices contributing to each entry of $F_k(M)$. For each sequence $U = u_0, u_1, \ldots, u_k$ of $k+1$ numbers chosen from $\{1, 2, \ldots, n\}$, form the k by k matrix whose (i, j) entry is the (u_i, u_{j+1}) entry of M. Let d_U be the determinant of this matrix, and let F_U be the n by n matrix that is zero except for d_U in the (u_0, u_k) position. We first make some observations about the F_U, and then will describe how to select a collection of sequences U such that $F_k(M)$ is the sum of those F_U.

First, note that if $u_1, \ldots, u_{k-1}$ contains the same number twice, then d_U is a determinant with either two equal rows, so $d_U = 0$ and F_U is the zero matrix. Sequences U that have such repetitions will be excluded. If one of these u_i is equal to u_0, the same argument shows that such F_U are zero matrices, but is is useful to allow some of these U to remain although they appear to contribute nothing,

If we interchange two of the numbers $u_1, \ldots, u_{k-1}$, then two rows of the determinant d_U are interchanged *and* two columns are interchanged. The two changes of sign leave us with the same determinant. We want to select only one representative of this class of sequences U that have the same value. For example, we can require that $u_1, \ldots, u_{k-1}$ be in increasing order. For each entry of $F_k(M)$, this gives $\frac{n!}{(k-1)!(n-k+1)!}$ matrices F_U associated with that position, although some of them are zero matrices. To show that this is the correct characterization of the $F_k(M)$, it is only necessary to verify that the steps (B), (T_k), and (M_k) preserve this characterization.

To verify (B), note that when $k = 1$, there is one U for each choice of u_0 and u_1, and F_U has the (u_0, u_1) element of M in the (u_0, u_1) position. Adding these builds M from its elements.

A sequence U contributes to a diagonal entry of $F_k(M)$ if $u_0 = u_k$. Those for which u_0 is equal to any other u_j have $F_U = 0$. For the others, the selected rows in computing d_U are just those whose index belongs to U, and the same is true of the selected columns. We have chosen to order rows and columns differently, but we could rearrange the columns to have the same order as the rows. This multiplies d_U by $(-1)^{k-1}$ and expresses the result as the determinant of a submatrix in which the indexing of rows and columns is the same. We could select standard representatives of a set $\{u_0, u_1, \ldots, u_{k-1}\}$ by arranging its elements in increasing order. The submatrices of M for such orderings are called **principal submatrices**. Each such set gives the same value of d_U independent of which element is selected to play the role of u_0 and u_k, but this value appears in k different F_U. The trace of $F_k(M)$ is thus k times $(-1)^{k-1}$ times the sum of the determinants of all principal submatrices.

Direct expansion of $\det(M - \lambda I)$ shows that these same determinants appear once in the coefficient of λ^{n-k} in the characteristic polynomial. There is also an alternation of sign. To show that the sign agrees with that chose for p_k, look at $k = 1$, where p_1 and the sum of the determinants of the 1 by 1 principal submatrices both give the trace of M. This proves that (T_k) gives the correct value of p_k if our description of $F_k(M)$ is correct.

When $k = 2$ in our original example, this description gives

$$-p_2 = \begin{vmatrix} 3 & 1 \\ 3 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 4 & 4 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 6 & 4 \end{vmatrix} = 6 - 8 + 6 = 4.$$

As a result of some experiments, it turns out that the individual submatrices whose determinants appear here should be modified by shifting the indexing of the columns. This will give a description of the entries all $F_k(M)$. Here is the rule:

Finally, consider (M_k). We must show that F_{k+1} is given by this description in terms of F_U as U runs through sequences $u_0, u_1, \ldots, u_{k+1}$ selected so that the sets $\{u_1, \ldots, u_k\}$ are all distinct. In the expansion of d_U by its first row, there is the (u_0, u_1) element of M multiplied by its cofactor. This cofactor is exactly the contribution to F_k associated with the sequence $u_1, \ldots, u_{k+1}$. Selecting different elements from $\{u_1, \ldots, u_k\}$

to play the role of u_1 doesn't change this cofactor relationship. There is also a term that contains the (u_0, u_k) element of M multiplied by its cofactor. This time, the cofactor is the plus or minus the determinant of the principal submatrix associated with $\{u_1, \ldots, u_k\}$. The sign is seen to be the opposite of the one in p_k.

The terms of the first type are parts of $MF_k(M)$ and those of the second type are part of $-p_k M$. Each part of the expansion of the determinant is identified with a unique expression in the quantity on the right side of (M_k).

This proof is still a little sketchy. It is included because it helps to explain the division by k in step (T_k). It also gives an explicit description of the matrices $F_k(M)$ that shows both that the p_k are the expressions arising from the expansion of the determinant defining the characteristic polynomial and that $F_{n+1}(M)$ is identically zero. In particular, this can be made to include a proof of both the Cayley-Hamilton Theorem and the Newton identities.

The division by k in step (T_k) introduces a concern that denominators may be introduced in finding the p_k. However, the combinatorial description shows that these denominators arise only to remove the redundancy that arises from counting the same expression k times in the trace of $F_k(M)$.

If M is invertible, $F_k(M)$ if a nonzero multiple of the identity, and is the product of M with $F_{k-1}(M) - p_k I$. The formula for the entries also relates $F_{k-1}(M) - p_k I$ to the adjugate of M if M is not invertible.

Conclusion. The value of this method of calculating characteristic polynomials is that the operations are easily performed, and numerical simplifications become available immediately. If the matrix M has integer entries, its characteristic polynomial will have integer entries, and only integers appear in its calculation.

The method is best for exact calculation, by hand or with a calculator, with matrices of modes size. I have done example by hand with $n = 5$. With a calculator, the main concern is that the entries of the $F_k(M)$ may become too large to be represented exactly since the growth is exponential in k. The space demands are modest since $F_k(M)$ may be discarded after $F_{k+1}(M)$ has been found.

If the entries are approximations to the real number entries of a matrix and the calculations are performed on a computer, there may be better methods for finding eigenvalues. However, if the goal is the computation of the coefficients of the characteristic polynomial, the algorithm may be useful. The additional information about the nature of the numbers appearing in the calculation will be useful in analyzing the algorithm.

Don't forget the exercises.