Math 642:550 — Summer 2003
MTTh 6:15–8:45 PM Hill 525
Prof. Bumby

Supplement 5:A robust method for finding characteristic polynomial

1. Introduction

It is easy to read too much into the definition of the characteristic polynomial as a determinant. As soon as the n by n matrix has $n > 3$, it becomes extremely difficult to evaluate that determinant reliably. None of the methods for evaluating determinants work well when the matrix is allowed to contain an indeterminate. We describe an alternative method that I have used to obtain reliable answers with hand computation for $n = 5$. It is also easy to program. A calculator version will be illustrated in lecture, and studies of the method from the middle of the twentieth century note that $n = 10$ was within reach on the digital computers of that time. The method is very well suited to cases, like integer matrices, that can be calculated exactly. An approximate method that concentrates on finding **some** eigenvectors exactly is likely to be more useful in practice than this method, which can find eigenvalues only as roots of the characteristic polynomial.

The need for such a method was impressed on me when I wanted to compute a 5 by 5 integer determinant by hand and every method I tried gave me a different answer.

More details on the method can be found in F. R. Gantmacher, *The Theory of Matrices*, Chelsea, 1990, § IV.5. This follows the description by D. K. Faddeev given in the 1940s. The method is attributed to Leverrier in 1840. In the 1940s, there were several independent discoveries of a more efficient version. When the authors learned that there discovery was more than a century old, attempts to expand the announcement to an article seemed to have been set aside. A few articles were published the gave further explorations of the method. The expositions in Alton S. Householder, *The Theory of Matrices in Numerical Analysis*, Dover, New York, 1975. (ISBN 0-486-61781-5) and J. S. Frame, "Matrix functions and applications", *IEEE Spectrum* **1** (1964) (five articles in numbers 3–7) have good lists of published rediscoveries.

We will find that Frame's proof that the algorithm is correct reverses history and includes easy proofs of the important results cited in the historical development.

2. Powers and traces

We begin by collecting some facts that are easily verified in the case when the characteristic polynomial has no repeated roots. The general case is also true, but the proof involves technical difficulties that are avoided in the modern proof.

The key to the method is that the sum of the elements on the diagonal of a square matrix M, called the **trace** of M is also equal to the sum of the eigenvalues of M. By itself, this isn't too useful. However, we also know that the eigenvalues of M^k are the k-th powers of the eigenvalues of M. Thus computing the traces of powers of M gives the sums of powers of the roots of the characteristic polynomial.

A small example will be useful. Let

$$M = \begin{bmatrix} \mathbf{3} & 1 & 5 \\ 3 & \mathbf{3} & 1 \\ 4 & 6 & \mathbf{4} \end{bmatrix}.$$

Then,

$$M^2 = \begin{bmatrix} \mathbf{32} & 36 & 36 \\ 22 & \mathbf{18} & 22 \\ 46 & 46 & \mathbf{42} \end{bmatrix} \qquad M^3 = \begin{bmatrix} \mathbf{348} & 356 & 340 \\ 208 & \mathbf{208} & 216 \\ 444 & 436 & \mathbf{444} \end{bmatrix}.$$

If the eigenvalues of M are denoted $\lambda_1, \lambda_2, \lambda_3$, then equating traces and power sums gives

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= 10 \\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 &= 92 \\ \lambda_1^3 + \lambda_2^3 + \lambda_3^3 &= 1000 \end{aligned}$$

Since M is recognized as 10 times a **Markov matrix**, we have $\sigma_1 = 10$, and without trying too hard, one finds $\lambda_3 = -\lambda_2$ and $2\sigma_2^2 = -8$. Thus, $\sigma_2 = 2i$. This gives the characteristic polynomial to be $\lambda^3 - 10\lambda^2 + 4\lambda - 40 = (\lambda - 10)(\lambda^2 + 4)$ — if the calculations are correct. Since we have the characteristic polynomial, we can verify the the other eigenvalues of the Markov matrix $(1/10)M$ are of absolute value less that 1 although, in this case they are purely imaginary.

3. The Newton identities

The coefficients of a polynomial are **elementary symmetric polynomials** of the roots found by expanding the product $\prod(\lambda - \lambda_i)$. There are relations between these elementary symmetric polynomials and the power sums used in our example that were put in a systematic form by Newton. The expressions take slightly different forms depending on how one chooses the signs of the coefficients. We choose to emphasize the coefficients of the polynomial rather than the symmetric functions. To be consistent with Faddeev, we write the characteristic polynomial in the form

$$\lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - \cdots - p_n,$$

with negative signs in front of all terms except the leading term, which has coefficient $+1$. In particular, the determinant turns out to be $(-1)^{n-1}p_n$. We also introduce s_k to stand for the sum of the k-th powers of the roots. The first few formulas are

$$\begin{aligned} p_1 &= s_1 \\ 2p_2 &= s_2 - p_1 s_1 \\ 3p_3 &= s_3 - p_1 s_2 - p_2 s_1 \end{aligned}$$

and, for each k,

$$kp_k = s_k - \sum \{\, p_i s_j : i > 0,\, j > 0,\, i + j = k \,\}$$

If these identities are used in order, then everything needed on the right side of an identity will be either a given s_n or a p_k obtained at a previous step. We will accept these identities on good authority. They follow from Frame's proof.

4. The Twentieth Century Improvement

Instead of applying Newton's identities after computing all the s_k, we construct the p_k and a sequence of matrices $F_k(M)$ such that

$$F_1(M) = M \tag{B}$$

$$p_k = \frac{1}{k}\operatorname{tr}\big(F_k(M)\big) \tag{T_k}$$

$$F_{k+1}(M) = M(F_k(M) - p_k I) \tag{M_k}$$

Here, (B) is the basis of the inductive construction. It tells us what $F_1(M)$ is. Then for $k = 1, 2, 3, \ldots$ we apply first (T_k) to use $F_k(M)$ to identify p_k and then (M_k) to determine $F_{k+1}(M)$. The only part of the algorithm that changes with k is the division by k in (T_k). This inductive definition is used to assure that

$$F_k(M) = M^k - p_1 M^{k-1} - p_2 M^{k-2} - \cdots - p_{k-1} M.$$

The Newton identities then show that (T_k) gives the correct value of p_k.

One nice feature of this algorithm is that it is self-checking. The Cayley-Hamilton theorem (mentioned in our text only in exercises — see the index) says that a matrix satisfies its characteristic polynomial. In the context of this method, this says that $F_{n+1}(M)$ must turn out to be identically zero. If you fail to obtain a zero matrix at this stage, then you know that you have made a mistake in your calculation. However, the operations in the algorithm consist only of modifying the diagonal of a matrix and multiplying by a fixed matrix. Frame gave a direct proof that $F_{n+1}(M)$ is zero, which will appear later in these notes, so that the Cayley-Hamilton Theorem appears as a consequence.

5. Revisiting the first example

Here is how the previous example looks when the newer method is used.

$$F_1(M) = \begin{bmatrix} \mathbf{3} & 1 & 5 \\ 3 & \mathbf{3} & 1 \\ 4 & 6 & \mathbf{4} \end{bmatrix} \tag{B}$$

$$p_1 = (3+3+4)/1 = 10 \tag{T_1}$$

$$F_2(M) = \begin{bmatrix} \mathbf{3} & 1 & 5 \\ 3 & \mathbf{3} & 1 \\ 4 & 6 & \mathbf{4} \end{bmatrix} \begin{bmatrix} \mathbf{-7} & 1 & 5 \\ 3 & \mathbf{-7} & 1 \\ 4 & 6 & \mathbf{-6} \end{bmatrix} = \begin{bmatrix} \mathbf{2} & 26 & -14 \\ -8 & \mathbf{-12} & 12 \\ 6 & -14 & \mathbf{2} \end{bmatrix} \tag{M_1}$$

$$p_2 = (2-12+2)/2 = -4 \tag{T_2}$$

$$F_3(M) = \begin{bmatrix} \mathbf{3} & 1 & 5 \\ 3 & \mathbf{3} & 1 \\ 4 & 6 & \mathbf{4} \end{bmatrix} \begin{bmatrix} \mathbf{6} & 26 & -14 \\ -8 & \mathbf{-8} & 12 \\ 6 & -14 & \mathbf{6} \end{bmatrix} = \begin{bmatrix} \mathbf{40} & 0 & 0 \\ 0 & \mathbf{40} & 0 \\ 0 & 0 & \mathbf{40} \end{bmatrix} \tag{M_2}$$

$$p_3 = (40+40+40)/3 = 40 \tag{T_3}$$

and $F_4(M) = 0$, as expected.

A convenient way to organize the work for hand computation is to arrange the n matrices $F_k(M)$ as the **second** row of a table, using the first row for the index k. The third row should contain the p_k, including the computation shown in the lines (T_k). A fourth row should contain the matrices $F_k(M) - p_k I$ in order to have them available to use as the second factor in the computation (M_{k+1}). While our description has emphasized the $F_k(M)$, the matrices in this fourth row have important applications. In particular, if M is invertible, $F_n(M)$ will be $\det(M)I$, so the previous matrix in the fourth row must be the **adjugate** of M. On the other hand, if M is singular, $F_n(M) = \mathbf{0}$, so the columns of the previous matrix in the fourth row belong to the **nullspace** of M. If it is not itself a zero matrix, its nonzero columns give eigenvectors of M for $\lambda = 0$.

6. Exercises

In order to check that you have organized the steps of this algorithm correctly, begin with some examples where you know the characteristic polynomial

1 A 3 by 3 identity matrix.

2 The 4 by 4 matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then try a more general 4 by 4 matrix.

3 The matrix

$$\begin{bmatrix} 3 & 1 & 5 & -2 \\ 3 & 3 & 0 & 1 \\ 4 & 6 & -4 & 3 \\ 2 & -1 & -2 & 0 \end{bmatrix}$$

7. A direct proof

The appearance of the adjugate in the fourth row of our computational scheme becomes a **central feature** of the method in Frame's approach. This approach identifies the bottom row of our table with coefficient matrices when the adjugate matrix of $\lambda I - M$ is written as a polynomial in λ with matrix coefficients.

The entries of the adjugate of $\lambda I - M$ are $(n-1)$ by $(n-1)$ subdeterminants of $\lambda I - M$, so each entry is a polynomial of degree at most $n-1$. Alternatively, it is a polynomial of degree $n-1$ with coefficients that are constant n by n matrices. Contrary to our previous convention, we write the adjugate of $\lambda I - M$ as

$$\sum_{j=0}^{n-1} \lambda^{n-1-j} B_j$$

with $B_0 = -I$. Indices of summation may be suppressed by adopting the convention that j ranges over **all** integers with $B_j = \mathbf{0}$ unless $0 \le j \le n-1$. Applying the same convention to the characteristic polynomial, calls for requiring $p_0 = -1$ and $p_j = 0$ unless $0 \le j \le n$.

Since this matrix is the adjugate of $\lambda I - M$,

$$(\lambda I - M)\Big(\sum \lambda^{n-1-j} B_j\Big) = f(\lambda) I$$

where $f(\lambda)$ is the characteristic polynomial of M. Equating the coefficients of λ^{n-k} on each side of this equation gives

$$B_k - MB_{k-1} = -p_k I.$$

To prove the algorithm correct **and** to prove the Cayley-Hamilton theorem (since we have agreed that $B_n = \mathbf{0}$), it suffices to verify that the trace of MB_{k-1} is kp_k for all k. This uses a more general result.

Lemma. *If $M(x)$ is any matrix function, then*

$$\frac{d}{dx}\det(M(x)) = \operatorname{tr}\left(\operatorname{adj}(M)\frac{d}{dx}M(x)\right)$$

Proof. Consider $\det(M(x))$ as a function of its entries $m_{i\,j}(x)$ and apply the chain rule to get

$$\frac{d}{dx}\det(M(x)) = \sum_{i,j}\frac{\partial}{\partial m_{i\,j}}\det\big(M(x)\big)\frac{d}{dx}m_{i\,j}(x).$$

The sum ranges over entries in the matrix. However, the partial derivatives of the determinant with respect to an entry $m_{i\,j}$ is just the **cofactor** of that entry. This is written in the (j,i) position in the adjugate, so the sum over j gives the (i,i) entry of the product in the statement of the lemma. Summing over i gives the trace of this matrix, as claimed.

To apply to the characteristic polynomial, note that the derivative of $\lambda I - M$ with respect to λ is the identity matrix. Thus

$$(k-n)p_k = \operatorname{tr} B_k = \operatorname{tr}(MB_{k-1}) - np_k.$$

This gives the expression for p_k in terms of B_{k-1} that was used in the algorithm.

8. Eigenvectors

Another use of the sequence of coefficients B_k is the **synthetic division** algorithm for finding the coefficients of a polynomial given in terms of powers of λ as a polynomial given in terms of powers of $\lambda - c$. The rule is to add c times the **new** B_{k-1} to B_k, for $k = 1, 2, 3, \ldots$.

To return to our example, the original sequence B_0, B_1, B_2 is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -7 & 1 & 5 \\ 3 & -7 & 1 \\ 4 & 6 & -6 \end{bmatrix} \begin{bmatrix} 6 & 26 & -14 \\ -8 & -8 & 12 \\ 6 & -14 & 6 \end{bmatrix}.$$

When expressed in terms of powers of $(\lambda - 10)$, this becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 5 \\ 3 & 3 & 1 \\ 4 & 6 & 4 \end{bmatrix} \begin{bmatrix} 36 & 36 & 36 \\ 22 & 22 & 22 \\ 46 & 46 & 46 \end{bmatrix}.$$

The last of these has columns that are eigenvectors, as predicted.

The same calculation can be done to translate by $2i$. The details a little messier, but the columns of the last matrix turn out to be multiples of $[1+i, -i, -1]^{\mathrm{T}}$.