Math 642:550 — Summer 2006
MTTh 6:00–8:30 PM Hill 705
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Supplement 5A, Matrix Exponentials Addendum

Section 8 (A triumph of abstraction) of the original supplement was too brief. It left the impression that the case of complex eigenvalues was the main goal, and all other cases would also involve trigonometric functions. This is so far from the truth that it needs to be corrected.

Nothing replaces the general conclusion that eigenvalues determine exponential functions appearing in the solutions and eigenvectors determine the numerical vectors of coefficients associated with those functions. The complex case is one of two cases where real solutions require functions other than pure exponentials. As noted in Section 3, the suggested technique is the **matrix equivalent** of completing the square for finding the eigenvalues. Initial exercises have been chosen to assure that the eigenvalues have the form $r + si$ with integers r and s, and the matrix J appearing in the solution have integer entries. These should be considered to be **accidental features** that are included to aid learning the method. This case is characterized by finding r such that $M - rI$ has **trace zero** and **positive determinant**. The determinant is s^2 and J is characterized by $M = rI + sJ$. You should **see** that $J^2 = -I$, and it is this property that is used in checking the you have a solution of the differential equation. The initial condition $Y(0) = I$ is also clear from this way of writing the solution because $e^0 = 1$, $\cos 0 = 1$, and $\sin 0 = 0$.

A modification deals with the case where $M - rI$ has **trace zero** and **negative determinant**. An answer can be written immediately simply by substituting the **hyperbolic functions** $\cosh x$ and $sinhx$ for the corresponding trigonometric functions. This answer can be converted to pure exponential functions using the **defining identities**

$$\cosh x = \frac{e^x + e^{-x}}{2} \text{ and } \sinh x = \frac{e^x - e^{-x}}{2}$$

The coefficients of the exponentials in e^{Mt} turn out to be the **projection matrices**

$$\frac{1}{2}(I + H) \text{ and } \frac{1}{2}(I - H).$$

If you choose to express the answer entirely in terms of exponentials, it should be checked in that form: the coefficient of each exponential should be a **rank one matrix**; the sum of the matrices should be I (to verify the initial condition); the product of the matrices should be the zero matrix (it is not necessary to check the product in both orders since the matrices will commute if their sum is I).

The case where $M - rI$ has **trace zero** and **zero determinant** is also troublesome in the standard approach. Repeated eigenvalues mean that there is **no basis of eigenvectors** unless M is a multiple of I (which forces $M - rI = \mathbf{0}$). When $M = rI + K$ with $K^2 = 0$, there are many ways to discover that

$$e^{Mt} = e^{rt}(I + Kt),$$

but knowing that you have the answer only requires seeing that this expression satisfies the differential equation and initial condition.

By emphasizing the determinant of a matrix of trace zero constructed from M, the details of the algorithm are pushed to the background and verification of the defining properties gets the spotlight. This is highly desirable: mistakes can occur anywhere, so you should not be dependent on your ability to remember the steps of an algorithm and perform them correctly. In a sense, this method only **suggests** what the answer should be and requires only that the definition of e^{Mt} be remembered and used for an (easy) check of the answer.

Once e^{Mt} has been found, it need only be multiplied on the right by any given $Y(0)$ to find the solution with those values. Again, verifying that the answer is correct is straightforward. The temptation to use a **formula** like (3) in the original supplement should be resisted. Such formulas are mainly developed to **guide the theory** and cannot be expected to lead to useful algorithms for finding the solution by either hand or machine computation.

Here is a **Maple** procedure that does all tests and writes the result.

An alternate approach to matrix exponentials in all 2 by 2 cases is to write a given matrix M in one of the forms: [Trig] $rI + sJ$ (with $\det J = 1$); [Hyp] $rI + sH$ (with $\det H = -1$; [Nil] $rI + K$ (with $\det K = 0$). In all cases, $r = (\operatorname{tr} M)/2$. Then, $\det(M - rI)$ is examined: a positive determinant gives [Trig] with $s = \sqrt{\det(M - rI)}$; a negative determinant gives [Hyp] with $s = \sqrt{-\det(M - rI)}$); a zero determinant gives [Nil]. A simple procedure keeps track of these tests.

```
FundamentalMatrix:=proc(M)::'Matrix'(2,2);
local r,s,sM,dM,HJK,m,n;
(m,n):=Dimension(M);
if (m<>2) or (n<>2)
    then error "This procedure only accepts 2 by 2 matrices"
    end if;
r:=Trace(M)/2;print('r'=r);
sM:=M-r;dM:=Determinant(sM);print('dM'=dM);
if (dM=0)
    then print("Nil case");
         return ScalarMultiply(ScalarMultiply(sM,t)+1,exp(r*t));
    elif (dM<0)
         then print("Hyp case");
         s:=sqrt(-dM);HJK:=sM.(1/s);
         return ScalarMultiply(cosh(s*t)+
                ScalarMultiply(HJK,sinh(s*t)),exp(r*t));
    else print("Trig case");
         s:=sqrt(dM);HJK:=sM.(1/s);
         return ScalarMultiply(cos(s*t)+
                ScalarMultiply(HJK,sin(s*t)),exp(r*t));
    end if;
end proc:
```

End of Supplement