Supplement 5, Matrix Exponentials

1. Introduction

It is now common for textbooks for a first course in Differential Equations to describe the solution of the first-order, linear, constant-coefficient system

$$\frac{dY}{dt} = MY, \tag{1}$$

where M is an n by n matrix, in terms of a matrix exponential e^{Mt}. However, there is only a limited discussion of techniques for finding the matrix exponential. The student is left with the impression that it is necessary to find eigenvalues and eigenvectors as the first step in employing this technique. This impression is reinforced by giving examples that are limited to $n = 2$ and a matrix M with integer entries whose characteristic polynomial factors to reveal distinct integer eigenvalues, for which the determination of eigenvalues is easy. We shall explore methods that give e^{Mt} directly. The computation may even be an efficient way to find the eigenvectors even when there is no particular need for the matrix exponential.

Examples may also given in which the eigenvalues are complex (usually Gaussian integers — $a + bi$ with a and b integers), and the solution is interpreted in this case. However, converting the solution to a useful form is tedious and rarely performed accurately. Thus, no interesting example can be solved quickly enough to be useful for lectures or examinations.

This appears to be an oversight, or perhaps **collective amnesia**, since there is **an easily remembered formula** (a phrase that was used as a title when part of this article was first used as course notes) for the solution. The uniqueness theorem shows that once any proposed answer is verified, it is the only correct answer. Thus, it is not necessary to perfect the underlying theory before proposing a method of solution. Any expression that is easily checked may be seen to solve the equation — however it was discovered. There is an advantage to having a obtaining solutions indirectly since it shifts the emphasis to verifying that the answer is correct rather than merely expecting the student to echo (some of) the steps in one method of computing the answer. In the same way that methods used in computer solutions are not just translations of a method used in proof, but are optimized for that environment, methods for hand computation should show human solvers the same respect.

Instead of trying to mechanize the process of solving differential equations, or the related process of indefinite integration, more emphasis should be given to a **guess and check** approach (which is usually called by the misleading name of "trial and error"). Although this makes the subject seem more of an art than a science, the results will sometimes allow **new principles** for **guessing answers** to be formulated. This approach will be used for $n = 2$ with complex eigenvalues. After finding the solution, a direct way of obtaining it will be explored. This approach will then be generalized to apply to examples with $n = 3$ and $n = 4$.

2. The matrix exponential

To begin, let us first ask: what is Y in (1)? The usual answer is, "a vector of n real (or complex) functions of t". An **initial condition** for (1) is a vector of n numbers which give Y when $t = 0$. It is convenient to consider Y as a vector function (rather than a vector **of** functions) and denote this initial value by $Y(0)$. The columns of e^{Mt} are then described as the solutions whose initial conditions are 1 in one coordinate and 0 in all others. Routine linear algebra says that once e^{Mt} is known, the solution of an initial value problem for the equation (1) is

$$Y(t) = e^{Mt}Y(0). \tag{2}$$

However, both (1) and (2) may be interpreted, **and are correct**, if Y is any matrix with n rows. Taking $Y(0)$ to be an identity matrix in (2) gives us the

Working Definition. *The matrix exponential* e^{Mt} *is the unique solution* $Y(t)$ *to* (1) *with* $Y(0) = I_n$*, the* n *by* n *identity matrix.*

3. The characteristic polynomial

Even the usual solution of (1) starts with a guess of the solution containing some parameters and then finds values of these parameters giving a solution. That guess is $Y = ve^{\lambda t}$, where v is a constant vector and λ is a number. Substituting into (1) shows that we have a solution if $Mv = \lambda v$. Since it is now common for students to meet linear algebra early in their studies, we may freely use the following

Standard Terminology. *The polynomial* $\det(M - \lambda I)$ *is called the characteristic polynomial of* M*; a zero of the characteristic polynomial is called an eigenvalue; the nonzero* v *with* $Mv = \lambda v$ *are called eigenvectors.*

When n distinct eigenvalues can be found, it is not difficult to find an eigenvector for each eigenvalue, and the n by n matrix S whose columns are the eigenvectors is an invertible matrix that satisfies $MS = S\Lambda$, where Λ is a diagonal matrix whose entries are the eigenvalues. This leads to

$$e^{Mt} = Se^{\Lambda t}S^{-1}, \tag{3}$$

which **looks like** a formula for the solution, and is easily verified to satisfy (1) and (2). However, implementing this formula requires that S and S^{-1} be computed, and this is not always easy (and is only possible if there is a basis of eigenvectors). Complex eigenvalues are possible and the algebra needed to find S and combine the factors in (3) is not familiar enough to have a high likelihood of leading to correct answers. Unless enough time is spent to gain fluency with linear algebra over $\mathbb{C}$, formula (3) is likely to be painful to use. It would be better to **refine our method of guessing**. Furthermore, an initial value problem formulated over the real numbers will have a solution that is a real function. It would be desirable to get such functions directly.

4. Complex exponentials

To interpret complex exponentials, the formula $e^{it} = \cos t + i \sin t$ is used. Differential Equations provide one justification for this formula. The derivative of $\cos t + i \sin t$ is the same expression multiplied by i, and the value of the expression at $t = 0$ is 1. Another justification, not really different, uses the Taylor series for exponential and trigonometric functions. Both depend on the ability to extend calculus to functions of a complex variable.

Suppose now that $n = 2$ and the eigenvalues of M are $r \pm si$ with $s \neq 0$. Then equation (3) applies. If the resulting expression is expanded and the complex exponentials $e^{(r \pm si)t}$ converted to real exponentials and trigonometric functions, then

$$e^{Mt} = Pe^{rt}\cos st + Qe^{rt}\sin st,$$

where P and Q are 2 by 2 matrices of real numbers. It turns out to be **much easier** to **identify** P and Q from this equation than to **calculate** them from (3). In particular, putting $t = 0$ in this expression leads (**immediately!**) to $P = I$. Note that this would not be as obvious if we had been looking at individual columns of e^{Mt} instead of the whole matrix.

To find Q, we can differentiate e^{Mt}. For the discovery of the solution, it suffices to consider only the value at $t = 0$. This should be M, and direct calculation shows it to be $rP + sQ$. Since we have $P = I$, knowing r and s allows us to obtain Q as $(1/s)(M - rI)$. The Cayley-Hamilton Theorem, which can be verified by direct computation for 2 by 2 matrices, shows that the square of Q is $-I$. A change to more suggestive notation gives

Theorem 1. *If $n = 2$ and M has eigenvalues $r \pm si$, then*

$$e^{Mt} = e^{rt}(I \cos st + J \sin st). \tag{E}$$

where I is the identity and J is characterized by $M = rI + sJ$. Furthermore, $J^2 = -I$ so that it plays the role of the number i in the algebra $\mathbb{R}[M]$ generated by M over $\mathbb{R}$.

For the special case in which $r = 0$, the derivative of $e^{sJt} = I \cos st + J \sin st$ is $-sI \sin st + sJ \cos st = sJe^{sJt}$, and the product rule extends this to the general case.

The characteristic polynomial of a 2 by 2 matrix M can be written as

$$\lambda^2 - \operatorname{tr}(M)\lambda + \det(M)$$

If the eigenvalues are complex, the roots $r + si$ are usually found by **completing the square**. Then, J is found by subtracting rI from M and dividing by s. We note the r is $\operatorname{tr}(M)/2$, so we could **begin this process** without solving the equation. Since the trace is a **linear function of matrices**, the matrix $M - rI$ has zero trace. For 2 by 2 matrices, this means that the diagonal entries are negatives of one another. This means that one should pause after finding it to verify this property. Since $\det(J) = 1$, we must have $\det(M - rI) = s^2$. For at least the matrices appearing in our exercises, it will be easier to find $\det(M - rI)$ than $\det(M)$. Moreover, the former is s^2 while the latter needs further processing by the **quadratic formula** or the **completing the square** process. In effect, finding $M - rI$ is the **matrix equivalent** of completing the square.

5. Example 1

Let

$$M = \begin{pmatrix} 7 & -13 \\ 2 & -3 \end{pmatrix}.$$

We have $\operatorname{tr}(M) = 4$, so $r = \operatorname{tr}(M)/2 = 2$, and we form

$$M - 2I = \begin{pmatrix} 5 & -13 \\ 2 & -5 \end{pmatrix}$$

to get a matrix of trace zero. It is easily seen that this has determinant 1, so it is J and $M = 2I + J$. Thus, the eigenvalues are $2 \pm i$ and

$$e^{Mt} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{2t} \cos t + \begin{pmatrix} 5 & -13 \\ 2 & -5 \end{pmatrix} e^{2t} \sin t.$$

Writing this as the product of e^{2t} and e^{Jt} simplifies the verification of the differential equation (1).

Note that the initial conditions (2) are clearly satisfied, and this verification was part of our discovery of the formula.

6. General properties of the matrix exponential

The key to finding e^{Mt} in Example 1 was to consider the relation between the exponentials of all matrices in $\mathbb{R}[M]$, to select J as the matrix whose exponential was easiest to find. The validity of this approach is based on the following analysis.

Lemma. *If $NM = MN$, then $Ne^{Mt} = e^{Mt}N$. Conversely, $Ne^{Mt} = e^{Mt}N$ implies that $NM = MN$.*

Proof. The first part **could be** proved using the power series representation of e^{Mt}, but it is more in keeping with the present approach to note that $F(t) = e^{Mt}N$ satisfies (1) and $F(0) = N$. Now, let $G(t) = Ne^{Mt}$. Clearly, $G(0) = N$ and $G'(t) = NMe^{Mt}$. Thus $G(t) = F(t)$ if and only if $G(t)$ satisfies (1), which holds if and only if $NM = MN$.

Corollary. *$e^{(A+B)t} = e^{At}e^{Bt}$ if and only if $AB = BA$.*

Proof. Since both $e^{(A+B)t}$ and $e^{At}e^{Bt}$ satisfy the same initial conditions, it suffices to show that $H(t) = e^{At}e^{Bt}$ satisfies (1) with $M = A + B$ if and only if $AB = BA$. We have $H'(t) = Ae^{At}e^{Bt} + e^{At}Be^{Bt}$, so $H'(t) = (A + B)H(t)$ is equivalent to $Be^{At} = e^{At}B$, which the lemma shows is equivalent to $AB = BA$

A more usual proof of the "if" part of this corollary generalizes the power-series proof of the corresponding result for the ordinary exponential. The series are multiplied term-by-term and terms are first grouped by the degree in t. The binomial theorem allows us to recognize the terms of degree n as $(A+B)^n t^n$, and $AB = BA$ is used to collect the terms in this proof.

Taking $A = rI$ and $B = sJ$ allows (E) to be recovered from special cases since scalar multiples of the identity commute with all matrices.

7. Stability

If $r > 0$, the solutions spiral outward; if $r < 0$, the solutions spiral inward; and if $r = 0$, they are closed curves (actually, ellipses). The local analysis of nonlinear equations shows that solutions with $r \neq 0$ behave in the same way as their linear part near the origin. In particular, if $r < 0$, the solutions are **stable** in the sense that solutions starting near the origin remain near the origin for all future time. In order to make this quantitative, it is useful to have a measurement of the distance to the origin that is sure to decrease on trajectories. Our formula for the solution in the linear case shows the ellipses traced by the solutions of the related equation $dY/dt = JY$ are the level sets of such a metric.

One could solve that equation and eliminate the parameter. This is not difficult, but the result can be simplified by introducing the matrix

$$J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Proposition. *The matrix J_0J is a symmetric matrix such that $J^{\mathsf{T}}(J_0J)J = J_0J$ and $I(J_0J)J + J^{\mathsf{T}}(J_0J)I = \mathbf{0}$. Thus, if $M = rI + sJ$, $M^{\mathsf{T}}(J_0J)M = (r^2 + s^2)J_0J$.*

Proof. Introducing names for the entries of J, and using the fact that it has trace zero. we have

$$J_0J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} c & -a \\ -a & -b \end{bmatrix}$$

verifying that the matrix is symmetric. Now,

$$J_0J = (J_0J)^{\mathsf{T}} = J^{\mathsf{T}}J_0^{\mathsf{T}} = -J^{\mathsf{T}}J_0.$$

Hence, $J^{\mathsf{T}}(J_0J)J = (J^{\mathsf{T}}J_0)(JJ) = (-J_0J)(-I) = J_0J$. The other claims are proved in a similar way.

Note that the definition of J shows that $-a^2 - bc = 1$, so $bc < 0$. Thus, exactly one of b or c is positive. If $c > 0$, corresponding to counterclockwise motion, the matrix J_0J will be positive definite.

This approach is still being developed, so no further details are available at this time.

8. More two by two matrices

For any 2 by 2 matrix M, if $M = rI + Q$, then $e^{Mt} = e^{rt}e^{Qt}$. We have seen that matrices Q of trace zero play a special role, and r can be chosen equal to $\operatorname{tr}(M)/2$, as in the complex case, to reduce to this special case. For the remainder of this section, Q will be assumed to be a 2 by 2 matrix of trace zero since the product rule completes the general proof once this case has been established.

Since $e^{Qt} = I\cos t + Q\sin t$ when $\det Q = 1$, we can **guess** that $e^{Qt} = I\cosh t + Q\sinh t$ when $\det Q = -1$, and this is easily verified. As before, other negative determinants are covered by taking a suitable constant multiple of t in this expression.

Nothing replaces the general conclusion that eigenvalues determine exponential functions appearing in the solutions and eigenvectors determine the numerical vectors of coefficients associated with those functions. The complex case requires functions other than pure exponentials, so it was important to have a way to identify those solutions without extensive algebra with complex numbers. As noted in Section 4, the suggested technique is the **matrix equivalent** of completing the square for finding the eigenvalues. Initial exercises have been chosen to assure that the eigenvalues have the form $r + si$ with integers r and s, and the matrix J appearing in the solution have integer entries. These should be considered to be **accidental features** that are included to aid learning the method. Additional exercises will be provided in which these arithmetic hints are not present.

This case of real eigenvalues is characterized by finding r such that $M - rI$ has **trace zero** and **negative determinant**. It is convenient to write $M - rI = sH$ with $\det H = -1$ in this case. The hyperbolic functions used by analogy to the case of complex eigenvalues can be expressed in terms of ordinary exponentials to recover the traditional solution by using the **defining identities**

$$\cosh x = \frac{e^x + e^{-x}}{2} \text{ and } \sinh x = \frac{e^x - e^{-x}}{2}$$

The coefficients of the exponentials in e^{Mt} turn out to be the **projection matrices** (though not necessarily **orthogonal** projection matrices).

$$\frac{1}{2}(I + H) \text{ and } \frac{1}{2}(I - H).$$

If you choose to express the answer entirely in terms of exponentials, it should be checked in that form: the coefficient of each exponential should be a **rank one matrix**; the sum of the matrices should be I (to verify the initial condition); the product of the matrices should be the zero matrix (it is not necessary to check the product in both orders since the matrices will commute if their sum is I).

The definition of e^{Qt} as a series also shows that $e^{Qt} = I + Qt$ when $\det Q = 0$. Again, however one guesses this solution, a proof consists of showing that it satisfies (1) and reduces to I when $t = 0$. This case where $M - rI$ has **trace zero** and **zero determinant** is most troublesome in the standard approach. Repeated eigenvalues mean that there is **no basis of eigenvectors** unless M is a multiple of I (which forces $M - rI = \mathbf{0}$). When $M = rI + K$ with $K^2 = 0$, there are many ways to discover that

$$e^{Mt} = e^{rt}(I + Kt),$$

but knowing that you have the answer only requires seeing that this expression satisfies the differential equation and initial condition.

By emphasizing the determinant of a matrix of trace zero constructed from M, the details of the algorithm are pushed to the background and verification of the defining properties gets the spotlight. This is highly desirable: mistakes can occur anywhere, so you should not be dependent on your ability to remember the steps of an algorithm and perform them correctly. In a sense, this method only **suggests** what the answer should be and requires only that the definition of e^{Mt} be remembered and used for an (easy) check of the answer.

This use of **leading special cases** seems much more robust than the traditional solution. Having derived the special case **once**, they are available for as long as you remember them (which should be a long time). Extension to other cases uses only the **addition formula** for the exponential when one of the factors is a scalar multiple of the identity matrix. Although this formula requires that the matrices appearing in it commute, this property holds in this case.

Since there are formulas covering all matrices of trace zero, the reduction to this case can be done before learning which case will apply.

Here is a MAPLE procedure that does all tests and writes the result.

Write the given matrix M in one of the forms: [Trig] $rI + sJ$ (with $\det J = 1$); [Hyp] $rI + sH$ (with $\det H = -1$; [Nil] $rI + K$ (with $\det K = 0$). In all cases, $r = (\operatorname{tr} M)/2$. Then, examine $\det(M - rI)$: a positive determinant gives [Trig] with $s = \sqrt{\det(M - rI)}$; a negative determinant gives [Hyp] with $s = \sqrt{-\det(M - rI)}$); a zero determinant gives [Nil]. A simple procedure keeps track of these tests.

```
FundamentalMatrix:=proc(M)::'Matrix'(2,2);
local r,s,sM,dM,HJK,m,n;
(m,n):=Dimension(M);
if (m<>2) or (n<>2)
    then error "This procedure only accepts 2 by 2 matrices"
    end if;
r:=Trace(M)/2;print('r'=r);
sM:=M-r;dM:=Determinant(sM);print('dM'=dM);
if (dM=0)
    then print("Nil case");
         return ScalarMultiply(ScalarMultiply(sM,t)+1,exp(r*t));
    elif (dM<0)
         then print("Hyp case");
         s:=sqrt(-dM);HJK:=sM.(1/s);
         return ScalarMultiply(cosh(s*t)+
                ScalarMultiply(HJK,sinh(s*t)),exp(r*t));
    else print("Trig case");
         s:=sqrt(dM);HJK:=sM.(1/s);
         return ScalarMultiply(cos(s*t)+
                ScalarMultiply(HJK,sin(s*t)),exp(r*t));
    end if;
end proc:
```

9. Laplace transforms

If a course includes Laplace transforms, they may be used to solve problems with given initial conditions. It is worth noting that this method applies to matrix solutions as well as the customary vector solutions. Thus $Y(s) = \mathcal{L}\{e^{Mt}\}$ can be found directly from

$$sY(s) - I = MY(s)$$

so that

$$Y(s) = (sI - M)^{-1}.$$

For 2 by 2 matrices, the ability to express the entries of the inverse of a matrix M directly in terms of the entries of M gives another approach to Theorem 1. Moreover, the reduction to the case of matrices of trace zero and the use of hyperbolic functions reflect familiar methods for efficiently recognizing inverse Laplace transforms.

10. A triumph of abstraction

By expressing the solution in terms of a matrix J (or H, or K) found in $\mathbb{R}[M]$, we have shifted emphasis from the matrix M to its corresponding linear transformation. That is, we are looking at the way that M acts on all vectors in $\mathbb{R}^n$ instead of emphasizing its action on one particular basis. This approach is also present in the use of eigenvectors, but the goal there only seems to find a better basis. The selection of the matrix J was based instead on abstract considerations — it represented the number i in $\mathbb{R}[M]$.

In fact, much more was shown. Only the fact that the minimal polynomial of M was of degree 2 was needed to obtain the expression for e^{Mt} and to verify that it was correct. This means that it is degree of the minimal polynomial rather than the size of the matrix that determines the structure of exponential. However, the use of matrices of trace zero makes essential use two dimensions, so higher dimensional examples may not share this feature. A description of $\mathbb{R}[M]$ using matrices that satisfy equations of low degree will lead to a simple computation of e^{Mt}.

11. A rank one example

Let

$$M = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \end{bmatrix}.$$

Then $M^2 = 6M$, so $(M - 3I)^2 = 9I$. Setting

$$J = \frac{1}{3}(M - 3I) = \frac{1}{3} \begin{bmatrix} -2 & 2 & 2 & 1 \\ 1 & -1 & 2 & 1 \\ 1 & 2 & -1 & 1 \\ 1 & 2 & 2 & -2 \end{bmatrix},$$

we are led to

$$e^{Mt} = e^{3t}(I \cosh 3t + J \sinh 3t) = \frac{1}{2}(I + J)e^{6t} + \frac{1}{2}(I - J).$$

This expression satisfies the correct initial conditions, and the verification that this satisfies the differential depends on the easily verified property that $MJ = M$. Note that J does **not** have trace zero. This feature, which seemed so important in the two by two case, depends on working with the characteristic polynomial, but this example is exploiting a **minimal** polynomial of low degree.

Note that this adds nothing to the previous computation of the eigenspaces of projections; its only purpose is to give an efficient description of e^{Mt}. Note, however, that

$$\frac{1}{2}(I + J) = \frac{1}{6}M \text{ and } \frac{1}{2}(I - J) = \frac{1}{6} \begin{bmatrix} 5 & -2 & -2 & -1 \\ -1 & 4 & -2 & -1 \\ -1 & -2 & 4 & -1 \\ -1 & -2 & -2 & 5 \end{bmatrix}.$$

These coefficient matrices are the projections onto the eigenspaces.

12. An example with repeated complex roots

This approach organizes the computation in cases that seem frightening with the traditional approach. For example, let

$$M = \begin{pmatrix} 2 & -2 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -5 & 1 \end{pmatrix}.$$

Since this is block-triangular, its characteristic polynomial is easily recognized to be $(\lambda^2 - 4\lambda + 8)^2$ (see exercise 34 in section 4.3). The eigenvalues are thus $2 \pm 2i$. This suggests that we first find

$$M^2 - 4M + 8I = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since this isn't zero, the characteristic polynomial is also the minimal polynomial. While $\lambda^2 - 4\lambda + 8$ can be factored into relatively prime polynomials over $\mathbb{C}$, there is no factorization into relatively prime polynomials with real coefficients. Since we are aiming to avoid algebra over the complex numbers, we seek a different approach.

The ring generated by M over $\mathbb{R}$ may be identified with the ring of polynomials in an indeterminate x modulo the primary ideal generated by $(x^2 - 4x + 8)^2$. Call this ring S. In S, the ideal T generated by $x^2 - 4x + 8$ is nilpotent, and S/T is isomorphic to $\mathbb{C}$. A key result for computation is that the ring homomorphism $S \to S/T$ has a left inverse. In particular, S has a subring isomorphic to $\mathbb{C}$, and elements of S can be written as a sum of a nilpotent element and an element of this subring. If M is written as a sum of an element of this subring and an element of T, e^{Mt} will be the product of the exponentials of two elements of S that satisfy equations of degree 2.

We will identify the subring isomorphic to $\mathbb{C}$ by producing an element of the form $j = (x-2)/2 + (x^2 - 4x + 8)y$, with $y \in S$ that plays the role of i in the sense that $j^2 = -1$ in S. Direct computation gives

$$j^2 + 1 \equiv \frac{1}{4}(x^2 - 4x + 8)(4xy - 8y + 1) \bmod (x^2 - 4x + 8)^2.$$

We get the value of j that we seek if $(x-2)y \equiv -1/4 \pmod{x^2 - 4x + 8}$. Since $(x-2)^2 \equiv -4 \pmod{x^2 - 4x + 8}$, the unique solution modulo $x^2 - 4x + 8$ is $y = (x-2)/16$. Computing the matrix corresponding to j gives

$$J = \frac{1}{8}\begin{pmatrix} 0 & -8 & 2 & 0 \\ 8 & 0 & 1 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & -20 & -4 \end{pmatrix}$$

Inverting the definition of J gives

$$M = 2 + 2J + N$$

with

$$N = -\frac{1}{8}(M-2)(M^2 - 4M + 8I) = \frac{1}{4}\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since J and N belong to $\mathbb{R}(M)$, $JN = NJ$ and we get

$$\begin{aligned} e^{Mt} &= e^{2t} e^{2Jt} e^{Nt} \\ &= e^{2t}(I \cos 2t + J \sin 2t)(I + Nt) \\ &= Ie^{2t} \cos 2t + Je^{2t} \sin 2t + Nt \cos 2t + JNt \sin 2t \end{aligned}$$

The matrix coefficients have all been shown except for

$$JN = \frac{1}{4}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{M^2 - 4M + 8I}{4}.$$

The general process of solving for y may have been obscured in the details of the special case. It is exactly Newton's Method in S (sometimes called Hensel's Lemma in this context). We are seeking a root of the separable polynomial $p(x) = x^2 + 1$, and if we have already found a root of $p(x)$ modulo T^k, then $p(x + y) \equiv p(x) + p'(x)y \pmod{T^{2k}}$. Since $p(x) \in T$ and $p'(x)$ is relatively prime to $p(x)$, $p'(x)$ is invertible modulo T. This allows us to find y such the $p(x + y) \in T^{2k}$. Since we are working in a ring for which some power of T is zero, iterating this leads to an exact solution.

13. Splitting by projections

We have described exponentials of matrices rather than of linear transformations, but the exponentials that we have found involved products of scalar functions of t with matrices in $\mathbb{R}[M]$. Such results could be expressed in a coordinate-free manner. However, matrices will be used in both proofs and examples, although the proofs will contain some matrices that need never be found in practice. This is because some constructions require transformations acting on a subspace of dimension m with $m < n$. To find an m by m matrix representing this action, one **chooses a basis** for the subspace. This basis is useful in the proof, but there will **usually not be a need to compute it**. (However, we will give an example where some invariant subspaces are easily found, so we can do most of the work with the action on those subspaces.) The n by n matrices that appear in an expression for e^{Mt} will all be found directly, and not in terms of any factorization that may be used in theoretical discussions.

The traditional solution when $n = 2$ and M has distinct real eigenvalues may be written in the form $e^{Mt} = E_1 e^{\lambda_1 t} + E_2 e^{\lambda_2 t}$ for some matrices E_1 and E_2. The **Spectral Decomposition** of M identifies E_i as the projection onto the λ_i-eigenspace of M whose kernel is the other eigenspace.

We have already seen a **multiplicative factorization** of the matrix exponential, but this is an **additive splitting**. With suitable modification, such a splitting can be found for any **idempotent** in $\mathbb{R}[M]$. Since basic expressions have been found that use spaces other than the one-dimensional space spanned by an eigenvector, it is more useful to have a means of using splittings inductively than to aim for a universal formula for the exponential.

To study this splitting, fix a matrix E such that $E^2 = E$ and $EM = ME$. Let m be the rank of E. Choose a basis for the column space of E and let B be an n by m matrix whose columns are this basis. Since M takes this column space to itself, there is a matrix M' such that $MB = BM'$. This M' is unique since the columns of B are linearly independent. Furthermore, since the columns of B are a basis for the columns space of E, there is a unique matrix B' such that $E = BB'$. Since $B = EB = (BB')B = B(B'B)$, the independence of the columns of B shows that $B'B = I$. Thus, $M^n E = B(M')^n B'$ for all n. More generally, for any polynomial p, $p(M)E = Bp(M')B'$. To get the corresponding result for matrix exponentials, we use the differential equation, as in previous results.

Proposition. *If $EM = ME$, $E^2 = E$, and $MB = BM'$, then*

$$e^{Mt}E = Be^{M't}B'. \tag{P}$$

Hence, if

$$e^{M't} = \sum_i f_i(t)p_i(M'),$$

then

$$e^{Mt}E = \sum_i f_i(t)p_i(M)E. \tag{S}$$

Proof. First, note that $e^{Mt}B$ and $Be^{M't}$ both satisfy (1) and evaluate to B when $t = 0$. The uniqueness theorem of differential equations then shows that they are equal. Now, multiply on the right by B' to obtain (P). The remaining statements follow from the discussion that preceded the statement of the Proposition.

If $ME = EM$ and $E^2 = E$, then $e^{Mt} = e^{Mt}E + e^{Mt}(I - E)$ and each of these terms is given by the action of M on the range of either E or $I - E$. The terms will be evaluated using the sum in (S) (or the corresponding statement for $I - E$). The matrices M' and B' and equation (P) are used in the proof, but do not need to be found. All that is needed is that the $e^{M't}$ have the form used in the proof, since this allows us to find the expression (S).

14. A three dimensional example

Markov matrices are a good source of examples allowing robust calculation, so let

$$M = \frac{1}{10}\begin{pmatrix} 3 & 1 & 5 \\ 3 & 3 & 1 \\ 4 & 6 & 4 \end{pmatrix}.$$

The column sums are all 1, so this has 1 as an eigenvalue, and its eigenvector is easily found to be $(18, 11, 23)$. The projection on this subspace that commutes with M is a matrix with all columns equal to the multiple of this vector with sum of entries equal to 1. Thus,

$$E = \frac{1}{52}\begin{pmatrix} 18 & 18 & 18 \\ 11 & 11 & 11 \\ 23 & 23 & 23 \end{pmatrix} \qquad I - E = \frac{1}{52}\begin{pmatrix} 34 & -18 & -18 \\ -11 & 41 & -11 \\ -23 & -23 & 29 \end{pmatrix}.$$

Since E is constructed from eigenvectors of M, $ME = E$. To get the action on the two dimensional space of vectors with column sums zero, which is the range of the projection $(I - E)$, form

$$M(I - E) = \frac{1}{260}\begin{pmatrix} -12 & -64 & 40 \\ 23 & 23 & -29 \\ -11 & 41 & -11 \end{pmatrix}.$$

This matrix has trace zero and one of its eigenvalues is known to be zero, so the sum of the other two eigenvalues is zero. The square of this matrix is seen to be $-(I - E)/25$, so $(I - E)M$ acts like $i/5$ on the range of $(I - E)$. Thus, using the known exponentials of the action of M on the subspaces, we have

$$e^{Mt} = \frac{e^t}{52}\begin{pmatrix} 18 & 18 & 18 \\ 11 & 11 & 11 \\ 23 & 23 & 23 \end{pmatrix} + \frac{\cos(t/5)}{52}\begin{pmatrix} 34 & -18 & -18 \\ -11 & 41 & -11 \\ -23 & -23 & 29 \end{pmatrix} + \frac{\sin(t/5)}{52}\begin{pmatrix} -12 & -64 & 40 \\ 23 & 23 & -29 \\ -11 & 41 & -11 \end{pmatrix}.$$

The product of M with the matrices in this expression have already been determined, so the verification that this is e^{Mt} is easy.

15. Expressing the projections

It was remarked in passing that the projection E commutes with M because it can be expressed as a polynomial in M. Although we found E in the Markov example by determining the eigenvector, an approach that applies to more general matrices would to compute the characteristic polynomial using the method of Leverrier. This method is very robust for hand computation, which led to its frequent rediscovery. If this characteristic polynomial can be factored into relatively prime factors, the Euclidean algorithm can be used to construct idempotents. In the Markov example, this gives $E = (25M^2 + I)/26$. Any factorization of the minimal polynomial of M into relatively prime factors reduces the determination of e^{Mt} to finding the exponentials of matrices whose minimal polynomials are those factors.

16. A triangular matrix example

Let

$$M = \begin{bmatrix} 5 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

We have previously (in supplement 4) found eigenvectors of M by row operations. The use of idempotents forms an interesting alternative. The eigenvalues are given by the diagonal entries 5, 3 and 2, so

$$M - 5I = \begin{bmatrix} 0 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{bmatrix}, M - 3I = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \text{ and } M - 2I = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

each send **one** of the eigenvectors to zero. Hence the product of **two** of these will send **two** of the eigenvectors to zero. Int the 3 by 3 case, there is only one more eigenvector and every vector is sent to a multiple of it. Thus, the resulting matrix is of rank 1, of the form uv^{T}. Since the matrix is also upper triangular, there is an index k such that u has zero in positions whose index is greater than k and v has zero in positions whose index is less than k. This requires that all **diagonal** positions, except in position k, be zero and that this value be the same as the number $v^{\mathsf{T}}u$. If each of these matrices is divided by this distinguished diagonal element, a family of matrices, each having zero diagonal except for 1 in one place. Such matrices can be shown to be idempotents whose sum is the identity. This gives

$$e^{Mt} = E_1 e^{\lambda_1 t} + E_2 e^{\lambda_2 t} + E_3 e^{\lambda_3 t}.$$

For this example, the products are

$$M_1 = \begin{bmatrix} 6 & -3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \qquad M_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{bmatrix}; \qquad M_3 = \begin{bmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

and the idempotents are found by dividing each of these by the nonzero number on its diagonal. The general formula gives

$$e^{Mt} = \frac{e^{5t}}{6} M_1 - \frac{e^{3t}}{2} M_2 + \frac{e^{2t}}{3} M_3 .$$

Furthermore, each idempotent is of rank 1, so any nonzero column can be taken as an eigenvector.

17. A more difficult triangular example

Let

$$M = \begin{bmatrix} 3 & -1 & 5 \\ 0 & -2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

This matrix has eigenvalues 3 and 2 with 3 repeated. In the course of our computation, we will discover that there is not a basis of eigenvectors. The simple eigenvalue −2 defines a useful splitting, and the method used in Section 14 shows that the projection on that space is

$$E_1 = \frac{1}{25}\left(M - 3I\right)^2 = \frac{1}{25}\begin{bmatrix} 0 & 5 & -2 \\ 0 & 25 & -10 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is easily seen to be an idempotent rank 1 matrix. The other idempotent is

$$E_2 = I - E_1 = \frac{1}{25}\begin{bmatrix} 25 & -5 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 25 \end{bmatrix}$$

and we find that

$$K_2 = \big(M - 3I \big) E_2 = \frac{1}{5}\begin{bmatrix} 0 & 0 & 23 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix E_2 projects onto a two dimensional subspace and K_2 is the nilpotent element in that space. It follows that

$$e^{Mt} = E_1 e^{-2t} + E_2 e^{3t} + K_2 t e^{3t}.$$

This expression is easily seen to satisfy the differential equation and initial condition, so **it must be right**. By aiming for the exponential directly instead of insisting on finding eigenvectors first, all difficulties of interpreting the repeated eigenvalue are avoided. Of course, we made things easier by starting with a triangular matrix, but the same process would work for any matrix after identifying its eigenvalues.

18. An example with a known splitting

The matrix

$$S = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

commutes with the matrix

$$R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Since $R^2 = I$, the matrices $E_+ = (I + R)/2$ and $E_- = (I - R)/2$ are idempotents whose sum is the identity. It is easily verified that

$$S\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad S\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}.$$

The 2 by 2 matrices tell us how that the action on the E_+ part has eigenvalues $(3 + \sqrt{5})/2$ and on the E_- part the eigenvalues are $(5 + \sqrt{5})/2$. If S is written as $2I - (1/2)R + (1/2)Q$, then Q will act like $\sqrt{5}$ on both parts. This gives

$$Q = \begin{bmatrix} 0 & -2 & 0 & 1 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 0 & -2 & 0 \end{bmatrix}$$

and

$$\begin{aligned} e^{St} &= E_+ e^{(3t/2)}\big(I\cosh(\sqrt{5}t/2) + Q\sinh(\sqrt{5}t/2)\big) + E_- e^{(5t/2)}\big(I\cosh(\sqrt{5}t/2) + Q\sinh(\sqrt{5}t/2)\big) \\ &= e^{2t}\big(I\cosh t/2 - R\sinh t/2\big)\big(I\cosh(\sqrt{5}t/2) + Q\sinh(\sqrt{5}t/2)\big) \end{aligned}$$

Since differentiating each factor in the last expression multiplies that factor by a known matrix, the effect of differentiating the product is easily found to be multiplication by S. This **multiplicative** factorization of e^{St} allows easy verification that this is the desired matrix exponential. The initial condition is clearly visible and the product rule reduces the verification of the differential equation to checking the derivatives of the factors.

19. Exercises

Find e^{Mt} for the following matrices M, and apply this to find the solution of $dv/dt = Mv$ with $v(0) = v_0$ for the given vector v_0.

We start with some **toy exercises** A–H, then move on to more general exercises U–X. In the toy exercises, although you will need to determine which case applies for the 2 by 2 matrices, after the trace zero matrix has been found by subtracting an appropriate multiple of the identity, its entries will be seen to have a common factor. Removing the common factor will leave a matrix whose determinant if 0, 1, or −1, allowing e^{Mt} to be written using integer matrices.

$$(A) \quad M = \begin{bmatrix} 25 & -30 \\ 15 & -17 \end{bmatrix}, v_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}; \qquad (B) \quad M = \begin{bmatrix} -7 & 14 \\ -10 & 17 \end{bmatrix}, v_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix};$$

$$(C) \quad M = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}, v_0 = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}; \qquad (D) \quad M = \begin{bmatrix} 5 & 24 & 33 \\ 0 & -3 & -10 \\ 0 & 0 & 2 \end{bmatrix}, v_0 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$(E) \quad M = \begin{bmatrix} -16 & 27 \\ -12 & 20 \end{bmatrix}, v_0 = \begin{bmatrix} 5 \\ -1 \end{bmatrix}; \qquad (F) \quad M = \begin{bmatrix} 23 & -12 \\ 30 & -13 \end{bmatrix}, v_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$(G) \quad M = \begin{bmatrix} -7 & 25 \\ -1 & 3 \end{bmatrix}, v_0 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}; \qquad (H) \quad M = \begin{bmatrix} 19 & -10 \\ 20 & -11 \end{bmatrix}, v_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Now, some more general matrices. The answers may involve fractions, square roots, or both, since they were not constructed from inter matrices in the role of H, J, or K. A symbolic form of the answer should be written: do not replace any number with a decimal approximation.

$$(U) \quad M = \begin{bmatrix} 5 & 4 \\ -1 & 3 \end{bmatrix}, v_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \qquad (V) \quad M = \begin{bmatrix} 7 & 2 \\ 2 & 3 \end{bmatrix}, v_0 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

$$(W) \quad M = \begin{bmatrix} 3 & 4 \\ 2 & 10 \end{bmatrix}, v_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}; \qquad (X) \quad M = \begin{bmatrix} 3 & -4 \\ 4 & 10 \end{bmatrix}, v_0 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

End of Supplement