Math 551, Assignment 3

Due Thursday, Oct. 5 in class

1. Let G be a group and S a subset of G such that $G = \langle S \rangle$. Suppose that T is a nonempty subset of G such that $sT \subseteq T$ for all $s \in S$. Show that $T = G$.

2. For all positive integers i, j, let $m_{ij} = 1$ or 3 if $|i - j| = 0$ or 1, respectively, and let $m_{ij} = 2$ otherwise. Show that for any $n \geq 1$, $\mathrm{gp}\langle t_1, \ldots, t_n \,|\, (t_i t_j)^{m_{ij}} = 1 \rangle \cong \Sigma_{n+1}$. (Hint. Let G be this group and H the subgroup generated by $t_1, \ldots, t_{n-1}$. Show that $t_n H \cup (t_{n-1} t_n) H \cup (t_{n-2} t_{n-1} t_n) H \cup \cdots \cup (t_1 \cdots t_n) H = G$, and bound $|H|$ by induction.)

3. Find all maximal subgroups of Σ_4 (i.e. those subgroups $H < \Sigma_4$ for which there are no subgroups K such that $H < K < \Sigma_4$). Sylow's Theorem is useful!

4. Suppose that $H \leq G$ and $|G : H|$ is finite. Show that there exists $K \lhd G$ such that $K \leq H$ and $|G : K|$ is finite.

5. Let $\phi : G \to H$ be a homomorphism. Let X and Y be subgroups of G. Prove that if $\ker \phi \leq X$, then $\phi(X \cap Y) = \phi(X) \cap \phi(Y)$. Show however that if the assumption on $\ker \phi$ is removed, the statement becomes false in general.

6. Let $\phi : G \to H$ be a homomorphism, and let $x \in G$. Show that if x has finite order, then so does $\phi(x)$, and the order of $\phi(x)$ divides that of x.

7. Let $H \lhd G$ and let $\pi_H : G \to G/H$ be the projection. Let p be a prime. Show that if P is a Sylow p-subgroup of G, then $\pi_H(P)$ and $P \cap H$ are Sylow p-subgroups of G/H and H, respectively. Show by example that $P \cap H$ is not necessarily a Sylow p-subgroup of H if the normality assumption on H is omitted.

8. Let $K \leq H \leq G$ and assume that $K \lhd H$. Let A be a subgroup of G. Show that $A \cap K \lhd A \cap H$, and $A \cap H / A \cap K$ is isomorphic to a subgroup of H/K.

9. Suppose that you are given presentations for two groups, say $X_1 = \mathrm{gp}\langle S_1 \,|\, R_1 \rangle$ and $X_2 = \mathrm{gp}\langle S_2 \,|\, R_2 \rangle$. Give presentations for

a) the free product $X_1 * X_2$ and

b) the direct product $X_1 \times X_2$.

Prove that your answers are correct.