Math 551, Assignment 5

Due Tuesday, October 24, in class

1. A group K is said to be complete if and only if the natural mapping $K \to \mathrm{Aut}(K)$ ($g \mapsto \mathrm{Int}(g)$ $\forall g \in K$) is an isomorphism. Show that if G is a group, $K \lhd G$ and K is complete, then $G = K \times H$ for some subgroup $H \leq G$.

2. Show that $SL_2(2)$ and Σ_4 are complete, but $PSL_2(\mathbf{R})$ and A_4 are not complete.

3. Show that $G = \mathbf{Z}_{16} \oplus \mathbf{Z}_2$ has a subgroup H such that $H \cong \mathbf{Z}_8$ and $G/H \cong \mathbf{Z}_4$.

4. In this problem, G is an abelian group. The subgroup $H \leq G$ is said to be *pure* in G if and only whenever $g \in G$, $n \in \mathbf{Z}$ and $ng \in H$, there is $h \in H$ such that $ng = nh$. The subgroup H is a direct summand of G if and only if $G = H \oplus K$ for some subgroup $K \leq G$. Show that

a) Any direct summand of G is pure in G.

b) If G has exponent p^n, p prime, and and $x \in G$ is an element of order p^n, then $\langle x \rangle$ is pure in G.

c) If H is pure in G, then any element of G/H of order m has a preimage in G which is also of order m.

d) If G is finite and H is a pure subgroup of G, then H is a direct summand of G.

5. For any group G, define the *derived series*

$$G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \cdots \geq G^{(n)} \geq \cdots$$

of G inductively by $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$ for each $i \geq 0$. Also set $G^{(\infty)} = \cap_{i=0}^{\infty} G^{(i)}$.

a) Show that G is solvable if and only if $G^{(n)} = 1$ for some n. (The least such n is called the derived length of G.)

b) Show that *for a finite group* G, the projection $G \to G/G^{(\infty)}$ is a universal homomorphism to a solvable group (in a sense which you are to formulate).

c) Construct a nonsolvable group G for which $G^{(\infty)} = 1$. (Hint. Try an infinite-dimensional version of the matrix group B.)

6. List all the characteristic subgroups (not just their isomorphism types!) of $\mathbf{Z}_{p^2} \oplus \mathbf{Z}_p$.

7. Let $G = G_1 \times G_2$, with $\rho_i : G \to G_i$ the natural projection, $i = 1, 2$. Given any subgroup $H \leq G$, show that $H \cap G_i \lhd \rho_i(H)$ for each i, and define, using H in a natural way, an isomorphism

$$\phi_H : \rho_1(H)/(H \cap G_1) \cong \rho_2(H)/(H \cap G_2).$$

Set up a one-to-one correspondence between the set of all subgroups of G and the set of all (ordered) quintuples $(A_1, B_1, A_2, B_2, \phi)$ such that $B_i \lhd A_i \leq G_i$ for each i, and $\phi : A_1/B_1 \to A_2/B_2$ is an isomorphism. Use your correspondence to count the number of non-abelian subgroups of $\Sigma_3 \times \Sigma_3$.