Due November 21, in class

Do 8 of these problems.
V is always a f.d.v.s. over a field F.

1. Let M be a torsion module ($M = T(M)$) over the PID R. For each prime $p \in R$ define $M[p^\infty] = \{m \in M \mid p^i m = 0 \text{ for some integer } i > 0\}$. Show that $M[p^\infty]$ is a submodule of M and

$$M = \bigoplus_p M[p^\infty],$$

the sum over a set of representatives for association classes of primes. (Note: there is no assumption that M is finitely generated.) $M[p^\infty]$ is called the p-primary part of M.)

2. A linear transformation $T : V \to V$ (or square matrix A) is nilpotent if and only if $T^n = 0$ (or $A^n = 0$) for some n. Show that if A is $m \times m$ and nilpotent, then $A^m = 0$.

3. Suppose that the square matrix A is in block diagonal form:

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & A_r \end{bmatrix}.$$

Show that $\nu(A) = \sum_{i=1}^r \nu(A_i)$, $\mu_A = \operatorname{lcm}_{i=1}^r \mu_{A_i}$ and $\chi_A = \prod_{i=1}^r \chi_{A_i}$. Which of these statements remain true if nonzero entries are allowed anywhere above the diagonal?

4. Suppose that A is an $n \times n$ matrix over a field F, and $\mu_A \in F[X]$ is irreducible of degree d. Show that d divides n.

5. How many pairwise nonsimilar 5×5 nilpotent complex matrices are there? How many pairwise nonsimilar 5×5 nilpotent real matrices are there?

6. A is a complex 5×5 matrix such that $\operatorname{Tr} A = 9$, $\det A = 16$ and $\mu_T(x) = (x-2)^2(x-1)$. What are the possible values for χ_T, and what are the possible Jordan canonical forms of A?

7. Compute the powers $J^2, J^3, \ldots,$ of $J = J(0,n)$. Then use $J(\alpha, n) = \alpha I + J$ to compute the powers of $J(\alpha, n)$. Finally, if A is an $m \times m$ complex matrix and $\det A \neq 0$, show that for each natural number n there is an $m \times m$ complex matrix B such that $B^n = A$.

8. Show that any complex square matrix is similar to its transpose.

9. Suppose that T and U are linear transformations from V to V such that $TU = UT$. Show that for every $\lambda \in F$, $\ker(T - \lambda)$ is invariant under U. If T and U are both diagonalizable, show that they are *simultaneously* diagonalizable, that is, there is a basis of V consisting of vectors which are eigenvectors of both T and U. Write the equivalent version of this statement for square matrices.

10. Suppose that A is a 2×2 integer matrix and $A^n = 1$ for some integer n. Show that $A^m = 1$ for some $m \leq 6$. Reason with eigenvalues, without doing any matrix multiplications.

11. Let A and B be real square matrices. If there exists a nonsingular complex matrix D such that $DAD^{-1} = B$, show that a nonsingular real matrix E can be found such that $EAE^{-1} = B$.

12. Suppose that $T : V \to V$ is a linear transformation. Show that $\mu_T = \chi_T$ if and only if there is a vector $v \in V$ such that V is spanned (as vector space) by the set $\{v, Tv, T^2v, \ldots, T^n v\}$. Furthermore, let $W \subseteq V$ be a T-invariant subspace, and set $U = T|W$. Show that if $\mu_T = \chi_T$, then $\mu_U = \chi_U$. (Hint. Don't use the first part. This is close to the theorem that subgroups of cyclic groups are cyclic.)

1. $M[p^\infty]$ is a submodule: Let $m, m' \in M[p^\infty]$ and $r \in R$. Then $p^i m = p^j m' = 0$ for some positive integers i, j, so $p^k(m + m') = 0$ where $k = \max(i,j)$; and $p^i(rm) = rp^i m = r0 = 0$ by the commutativity of R.

M is generated by the $M[p^\infty]$: Let $m \in M$. Then Rm is a submodule of M, and the mapping ${}_RR \to Rm$, $r \mapsto rm$, is a surjective homomorphism of R-modules. The kernel is of the form Rr for some $r \in R$ as R is a PID. Therefore $Rm \cong R/Rr$. Factorizing $r = up_1^{a_1} \cdots p_n^{a_n}$ with u a unit we have $Rr = Rur$ so

$$Rm \cong {}_RR/Rp_1^{a_1} \cdots p_n^{a_n} \cong {}_RR/Rp_1^{a_1} \oplus \cdots \oplus {}_RR/Rp_n^{a_n}$$

by the Chinese Remainder Theorem. Therefore Rm is the direct sum $Rm = M_1 \oplus \cdots M_n$ of submodules $M_i \cong {}_RR/Rp_i^{a_i}$. Clearly $M_i \subseteq M[p_i^\infty]$ so $Rm \subseteq \sum_p M[p^\infty]$. As m was arbitrary, $M = \sum_p M[p^\infty]$.

The sum of the $M[p^\infty]$ is direct: From the definition of direct sum, it must be verified that for any p, $M[p^\infty] \cap \sum_{q \neq p} M[q^\infty] = 0$. Suppose that $m \in M[p^\infty] \cap \sum_{q \neq p} M[q^\infty]$. Then $m = \sum_{i=1}^n m_i$ where for each i, $m_i \in M[q_i^\infty]$, and the primes $p, q_1, \ldots, q_n$ are pairwise non-associated. Moreover $p^k m = 0$ and $q_i^{k_i} m_i = 0$ for suitable positive integers k, k_i. Let $Q = q_1^{k_1} \cdots q_n^{k_n}$. Then $Qm_i = 0$ for all i so $Qm = 0$. Also $p^k m = 0$. But $\gcd(p^k, Q) = 1$ so we may write $1 = ap^k + bQ$ for some $a, b \in R$, and then $m = ap^k m + bQm = 0$. QED

2. By nilpotence, $\mu_A(X)$ divides a power of X so $\mu_A(X) = X^r$ for some r. But μ_A divides χ_A, which has degree n, so $r \leq n$. Therefore $A^n = 0$. QED

4. Let $f_1 | f_2 | \cdots | f_r$ be the elementary divisors of A. We know that $\mu_A = f_r$ and $\chi_A = f_1 \cdots f_r$. Since $f_r = \mu_A$ is irreducible, $f_i = f_r$ for all i and so $\chi_A = \mu_A^r$. The degree of χ_A is n, so $n = dr$. QED

5. We can use Jordan canonical form to determine similarity classes over $\mathbf{C}$. Over $\mathbf{R}$, which is not algebraically closed, we can't always do this, but we can always use rational canonical form. Let A be a nilpotent real or complex 5×5 matrix. The minimal polynomial of A is then X^k for some i. The elementary divisors of A are $f_1 | f_2 | \cdots | f_r = X^k$ so $f_i = X^{k_i}$ for each i, where $k_1 \leq k_2 \leq \cdots \leq k_r = k$. Since $\chi_A = f_1 \cdots f_r$ has degree 5, $\sum k_i = 5$. The rational canonical form of A therefore is a block-diagonal matrix whose blocks are the companion matrices of $X^{k_1}, \ldots, X^{k_r}$. (These companion matrices are actually the Jordan blocks $J(0, k_i)$.) So the number of similarity classes is the number of partitions of 5, which is 7.

6. Too much information is given. From the given, we know that the characteristic polynomial of A is $(X-2)^a(X-1)^b$ (since μ_A and χ_A have the same irreducible factors), where $a + b = 5$, and $a \geq 2$, $b \geq 1$ (since $\mu_A | \chi_A$). Thus the five eigenvalues are two, three or four 2's, and the rest 1's. We also know that $\det A$ is the product of the eigenvalues and $\operatorname{Tr} A$ is the sum of the eigenvalues. Therefore the eigenvalues are four 2's and one 1. The total size of the Jordan blocks for each eigenvalue is the multiplicity of that eigenvalue, and μ_A is obtained from the Jordan blocks of maximal size for each eigenvalue. Therefore A has Jordan canonical form $(J(1,1), J(2,2), J(2,2))$ or $(J(1,1), J(2,2), J(2,1), J(2,1))$.

8. We have $A = DJD^{-1}$ for some invertible D and some J in Jordan canonical form. Therefore $A^T = EJ^TE^{-1}$ where $E = (D^T)^{-1}$. So it is enough to show that J is similar to its transpose. This easily reduces to the case of a single Jordan block. So it is enough to show that the Jordan canonical form of $J(\alpha, n)^T$ is $J(\alpha, n)$. One easily calculates that $(J(\alpha, n)^T - \alpha I)^n = 0$ but $(J(\alpha, n)^T - \alpha I)^n = 0$, so $J(\alpha, n)^T$ has minimal polynomial $(X - \alpha)^n$. As $J(\alpha, n)^T$ is $n \times n$, its Jordan canonical form is therefore $J(\alpha, n)$. QED

9. Lemma: If $U : V \to V$ is a diagonalizable linear transformation and W is a U-invariant subspace, then $U|W$ is diagonalizable. Proof: U is diagonalizable so μ_U is square-free. But $\mu_U(U|W) = 0$ so $\mu_{U|W}$ divides μ_U, hence is square-free too, so $U|W$ is diagonalizable.

Given that $TU = UT$, if $v \in \ker(T - \lambda)$ for some λ, then $Tv = \lambda v$, so $T(Uv) = U(Tv) = U(\lambda v) = \lambda(Uv)$ and thus $Uv \in \ker(T - \lambda)$. So $\ker(T - \lambda)$ is U-invariant. Since T is diagonalizable, V is the direct sum $V = \bigoplus_\lambda \ker(T - \lambda)$, the sum over all eigenvalues. For each λ, $U|\ker(T - \lambda)$ is diagonalizable so the eigenspace $\ker(T - \lambda)$ has a basis B_λ consisting of eigenvectors for U. Then $\cup_\lambda B_\lambda$ is a basis of V consisting of simultaneous eigenvectors for T and U.

The matrix-theoretic statement is: If A and B are square matrices such that A and B are diagonalizable and $AB = BA$, then there exists invertible D such that DAD^{-1} and DBD^{-1} are both diagonal.

11. Write $D = D_1 + iD_2$ where D_1 and D_2 are both real matrices. $DAD^{-1} = B$ implies $DA = BD$. Equate real and imaginary parts of both sides to get $D_1A = BD_1$ and $D_2A = BD_2$ (since A and B are real). If D_i is invertible for either $i = 1$ or 2, we are done. In general set $D(t) = D_1 + tD_2$ (t an indeterminate). Then $D(t)A = BD(t)$. This equation remains true when we specialize t to be any particular complex number. It suffices to find $t \in \mathbf{R}$ such that $\det D(t) \neq 0$, for then $D(t)AD(t)^{-1} = B$. But $f(t) = \det D(t) \in \mathbf{R}[t]$, and $f(t)$ is not the zero polynomial because $f(i) = \det D(i) = \det D \neq 0$. Therefore $f(t) \neq 0$ for some (infinitely many!) $t \in \mathbf{R}$.