Math 551, Assignment 7, due Monday, December 2, in my mailbox

Throughout this assignment, V is a finite-dimensional vector space over the field k, and $B = \{v_1, \ldots, v_n\}$ is an ordered basis of V.

1. Let $\mathcal{B}$ be a bilinear form on V. Let β_L, β_R be the associated mappings from V to V^*. Show that

a) $\beta_L = \beta_R^*$

b) $[\beta_R]_{B^*}^{B} = \{\mathcal{B}\}_B$. (Or is it $[\beta_L]_{B^*}^{B}$?)

c) The set of all matrices of the form $\{\mathcal{B}\}_B$, as B varies over all the ordered bases of V (but $\mathcal{B}$ remains fixed) is an equivalence class under the equivalence relation $\approx$ defined by $A \approx A'$ if and only if $A' = P^T AP$ for some invertible P.

2. Suppose that $T : V \to V$ is a diagonalizable linear transformation. Let $W \subseteq V$ be a subspace such that $T(W) \subseteq W$. Thus T induces linear transformations $T|W : W \to W$ and $T|^{V/W} : V/W \to V/W$, which you are asked to prove are both diagonalizable as well. (Hint. Relate their minimal polynomials to μ_T.)

3. Suppose that $T : V \to V$ and $U : V \to V$ are linear transformations such that

$$TU = UT.$$

a) Show that for any eigenspace W of T, or indeed for any generalized eigenspace W of T, $U(W) \subseteq W$.

b) Suppose that T and U are both diagonalizable (that is, there is a basis of V whose elements are eigenvectors for T, and there is also a basis whose elements are eigenvectors for U). Show that they are simultaneously diagonalizable, that is, there is a basis of V whose elements are eigenvectors both for T and for U. (Use the previous problem.)

c) Interpret b) as a theorem about matrices.

4. A lattice Λ in $\mathbf{R}^n$ is defined to be an abelian subgroup of $\mathbf{R}^n$ (under addition) which is generated by an $\mathbf{R}$-basis B of $\mathbf{R}^n$.

a) Show that Λ is then a free abelian group with basis $B = \{v_1, \ldots, v_n\}$.

The lattice Λ is said to be integral if and only if $(v, w) \in \mathbf{Z}$ for all $v, w \in \Lambda$. Here $(\cdot, \cdot)$ is the standard Euclidean inner product. The lattice dual of Λ is defined to be the group

$$\Lambda^* = \{v \in \mathbf{R}^n \mid (v, w) \in \mathbf{Z} \text{ for all } w \in \Lambda\}.$$

b) If Λ is integral, show that $\Lambda \subseteq \Lambda^*$, that

$$|\Lambda^*/\Lambda| = \big|\det[(v_i, v_j)]_{i,j=1}^n\big|,$$

and that the structure of Λ^*/Λ can be revealed by applying suitable integer row and column operations to the matrix $[(v_i, v_j)]_{i,j=1}^n$.

(Hint. Let $w_1, \ldots, w_n \in \mathbf{R}^n$ be such that $(v_i, w_j) = \delta_{ij}$ (why do these vectors exist?) and show that Λ^* is a lattice with basis $\{w_1, \ldots, w_n\}$. Then see the previous assignment.)

c) What is the structure of the group Λ^*/Λ in the following cases?

(i) $B = \{[1 \quad -1 \quad 0], [0 \quad 1 \quad -1]\}$

(ii) $B = \{[1 \quad -1 \quad 0 \quad 0], [0 \quad 1 \quad -1 \quad 0], [0 \quad 0 \quad 1 \quad -1], [0 \quad 0 \quad 1 \quad 1]\}$.