Mathematics 551 **Algebra** **Fall 2006**

Week 5 Normal subgroups, Sylow theorems, Composition Series
Sections 2.5-2.8

1. Suppose that the faces of a regular icosahedron are to be colored red, blue, green or yellow. How many different possible colorings are there if two colorings are regarded the same if one is obtained from the other by a rotation of the icosahedron. You may wish to use that the group of rotational symmetries of the icosahedron is the alternating group on 5 letters.

2. Prove the following statements using group actions.

   a) Suppose that H is a subgroup of G with finite index n. Show that there is a normal subgroup K of G of index dividing $n!$ which is contained in H.

   b) Suppose that H_1, H_2 are subgroups in a group G. Suppose that H_1 has finite index in G. Show that $H_1 \cap H_2$ has finite index in H_2. Use this to show that the intersection of subgroups of G of finite index also has finite index in G.

3. Let $n > 1$ be an integer.

   a) If H is a subgroup of a finite abelian group G then G has a subgroup isomorphic to G/H.

   b) Prove or disprove: There is a cylic subgroup of $\mathbf{Z}/n^3\mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z}$ with cyclic quotient of order n^2.

4. Hungerford II.5.9

5. Hungerford II.6.1

6. Hungerford II.6.9

7. Hungerford II.8.1

8. Hungerford II.8.9