Takehome Exam – Algebra I (551)

Prof. Weibel
Math 551 (Fall 2008)
due Tuesday, December 2, 2008

In this exam, F is a field.

1. Let R be a principal ideal domain, and $n \geq 3$.
 (a) If i, j, k are distinct indices, compute $[E_{ij}(r), E_{jk}(s)]$ in $SL_n(R)$.
 (b) Prove that $SL_n(R)$ is a perfect group. (Recall that G is *perfect* if $G = [G, G]$.)

2. (From Fall 2008 WQ)
 Suppose that A and B are diagonalizable matrices in $M_n(F)$. Prove that A and B commute if and only if there is a basis of F^n whose elements are eigenvectors of *both* A and B.

3. (From Fall 2008 WQ)
 Let C denote the set of conjugacy classes in $G = GL_5(F)$ and let $D \subset C$ denote the set of conjugacy classes of matrices $A \in G$ whose minimal polynomial is $f(x) = (x-1)^2(x+1)$. Determine the number $|D|$ and find representatives for each conjugacy class in D. Treat the case of characteristic 2 separately.

4. (From Spring 2008 WQ)
 Consider the free abelian group $\mathbb{Z}^4$ on 4 generators, and let K be the subgroup of $\mathbb{Z}^4$ generated by the elements

$$(5, -2, -4, 1), \quad (-5, 4, 4, 1), \quad (0, 6, 0, 6).$$

 Determine the structure of the abelian group $\mathbb{Z}^4/K$ as a direct product of cyclic groups. No bullying allowed!

5. (From Spring 2008 WQ)
 Let $A = (a_{ij})$ be a matrix in $M_n(\mathbb{C})$. Prove that if λ is an eigenvalue of A then there is an i, $1 \leq i \leq n$, such that

$$|\lambda| \leq \sum_{j=1}^{n} |a_{ij}|$$