Homework solution - Algebra II (552)

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Let E/F be a finite separable field extension, and D a division algebra over F.
Problem: Show that $A = E \otimes_F D$ is semisimple.

Observation. *Since $E = F[x]/(f)$, E is a vector space with basis $\{1, x, x^2, ..., x^n\}$ and hence A is a left vector space over D with basis $\{1 \otimes 1, ..., x^i \otimes 1, ...\}$. Every element of A has a unique expansion $\sum x^i \otimes d_i$ in terms of this basis.*

Reduction. *We may assume that the center of D is F.*

To see this, suppose that the center of D is a field K. Since $E \otimes_F K$ is semisimple, it has the form $\prod K_i$ for separable field extensions K_i of K. Then

$$E \otimes_F D = (E \otimes_F K) \otimes_K D = \prod K_i \otimes_K D.$$

If each $K_i \otimes_K D$ is semisimple, so is $E \otimes_F D$.

Proof. We may assume that F is the center of D. In this case, we will show that $A = E \otimes_F D$ is simple. If not, there is a nonzero ideal I of A. Pick a nonzero element $a = \sum x^i \otimes d_i$ with the smallest number of nonzero d_i. If the smallest power of x that appears is x^e, we can replace a by $x^{-e}a$ to assume that $d_0 \neq 0$. Replacing a by ad_0^{-1}, we can even assume that $d_0 = 1$.
For each $d \in D$, the element $da - ad$ belongs to I, and the coefficient of x^i is $dd_i - d_i d$. Since the coefficient of x^0 is zero, it has a smaller number of nonzero terms than a — and so must be zero. Thus each d_i is in the center of D, i.e., $d_i \in F$. But then $a \in I$ is a nonzero element of $E \otimes_F F = E$. Since E is a field, this implies that I contains the unit $1 \otimes 1$ and hence that $I = A$. This proves that A is a simple ring, as claimed.