Problem Set 1.

1. Find a monic polynomial with integer coefficients which has $\sqrt{5}+\sqrt{7}$ as a root.
2. Show that given integers $r_1 \geq 0, r_2 \geq 0, r_1+r_2 \geq 1$ there exists a number field K such that $K \otimes_{\mathbf{Q}} \mathbf{R} = \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$.
3. Let D be a square free integer (not equal to 0 or 1). Find the maximal order and discriminant of the number field $\mathbf{Q}(\sqrt{D})$. Determine all orders in this number field.
4. (Dedekind 1878) Let K be the number field $\mathbf{Q}(\alpha)$, where α is a root of $x^3+x^2-2x+8=0$. Show that the maximal order of K is $\mathbf{Z}+\mathbf{Z}\alpha+\mathbf{Z}\frac{\alpha+\alpha^2}{2}$. Compute the discriminant of K.
5. Let p be a prime number, let ρ be a primitive p^{th} root of unity, and let K be the number field $\mathbf{Q}(\rho)$.
   a) Show that $\mathbf{Z}[\rho]$ is an order in K with discriminant $(-1)^{(p-1)(p-2)/2}p^{p-2}$.
   b) Show that any quadratic number field K is contained in the field generated by roots of unity of order $|d_K|$. Hint: Show this for $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{p})$ for odd primes p and take compositums of fields.

      Remark: It is a theorem of Kronecker and Weber that any number field K which is a Galois extension of $\mathbf{Q}$ with abelian Galois group is a subfield of the field generated by the roots of unity of order $|d_K|$.