

Problem Set 8.

1. (Chinese Remainder Theorem) Let R be a commutative ring with identity and let A_1, \dots, A_n be ideals of R such that $A_i + A_j = R$ when $i \neq j$. Show that given x_1, \dots, x_n in R , there exists x in R such that $x - x_i \in A_i$.
2. Suppose that R is a commutative ring with identity in which all nonzero prime ideals are maximal and every nonzero ideal is equal to a product of powers of prime ideals. If R has finitely many prime ideals, use the Chinese remainder Theorem to show that every ideal of R is a principal ideal (note that R is not necessarily a domain).
3. Let R be a Dedekind domain. Let I be a nonzero ideal of R and let $a \in I$ be a nonzero element. Show that there exists $b \in I$ such that $I = aR + bR$, and hence every ideal of a Dedekind domain is generated by 2 elements. (Hint: Apply (2.) to $R/(a)$.)
 - b) Let \mathbf{O} be an order in a number field. Let I be an ideal in this order of index prime to the index of \mathbf{O} in the maximal order. Show that I is generated by 2 elements as an \mathbf{O} module.
 - c) Find an order \mathbf{O} in a number field and an ideal in the order that is not generated by 2 elements as an \mathbf{O} module.
4. Let K be a number field and let $Cl(K)$ be the ideal class group of invertible fractional ideals modulo principal fractional ideals. Let J be a nonzero ideal in the ring of integers of K . Show that every class in the class group contains a representative prime to J by using the Chinese Remainder Theorem to construct for each nonzero ideal I an element $a \in I$ with $(a)I^{-1}$ relatively prime to J and investigate what this implies for ideal classes.