

MATHEMATICS 642:573—NUMERICAL ANALYSIS  
MIDTERM PROBLEM SHEET

1. The following is a divided-difference table for a function  $f(x)$ .

$x$	$f(x)$	$f[ , ]$	$f[ , , ]$	$f[ , , , ]$	$f[ , , , , ]$	$f[ , , , , , ]$
0.0	0.0	1.221403	1.352109	0.74840	0.276159	0.0764401
0.2	0.2442806	1.7622465	1.80115	0.9693312	0.352600	
0.4	0.5967299	2.482707	2.38275	1.2514104		
0.6	1.0932713	3.435007	3.13360			
0.8	1.7804327	4.6892455				
1.0	2.7182818					

- (A) Based on this information, does  $f(x)$  appear to be a polynomial of degree  $\leq 4$ ?
- (B) Write a polynomial of minimal degree interpolating  $f(x)$  at all the values for which it is tabulated above. {Use the Newton form!}
- (C) Suppose you had only the information in the table that comes from the function values at  $x = 0.0, 0.2$  and  $0.4$ . Give an estimate for  $f'(0.1)$  using all that information, and justify your answer.

2. The following is a divided-difference table for a function  $f(x)$ .

$x$	$f(x)$	$f[ , ]$	$f[ , , ]$	$f[ , , , ]$	$f[ , , , , ]$	$f[ , , , , , ]$
-1.0	-2.0	2.3056	-2.64	2.0	-1.0	1.0
-0.6	-1.07776	0.1936	-0.24	0.4	1.0	
-0.2	-1.00032	0.0016	0.24	2.0		
0.2	-0.99968	0.1936	2.64			
0.6	-0.92224	2.3056				
1.0	0.0					

- (A) On the basis of the information contained in this table, give a lower estimate of the minimum distance  $\rho_4(f)$  between  $f$  and the polynomials  $\mathcal{P}_4$  of degree  $\leq 4$ , where distance is measured in the uniform norm  $\| \cdot \|_\infty$  of the space  $\mathcal{C}([-1, 1])$  of continuous real-valued functions on the interval  $[-1, 1]$ .
- (B) Knowing the Chebyshev polynomials of degrees  $\leq 3$  from Atkinson, p. 211, and knowing the recursion, find  $T_4(x)$  and  $T_5(x)$  (explicitly, in power form).
- (C) If one knows that the function in the table is  $f(x) = x^5 - 1$ , what is the value of  $\rho_4(f)$ ? Write out, explicitly, the polynomial of degree  $\leq 4$  for which the minimum distance is attained.

3. Consider the system of nonlinear equations

$$\begin{aligned}x^2 + y - 12 &= 0 \\x - y^2 + 6 &= 0.\end{aligned}$$

Suppose one wishes to find a solution of this system by an iterative technique, with the initial approximation  $(x, y) = (2, 4)$ .

- (A) Give two iterative techniques for solving this nonlinear system: one should be related to Newton's method, while the other method should find the fixed point of a (different, and preferably first-order) contraction mapping. Describe a region of  $(x, y)$ -space in which the mapping for which you seek a fixed point will actually be a contraction; make the region as large as you can.
- (B) Find the result of one iteration of each method you give, beginning at the initial approximation  $(2, 4)$ . {The nearest exact solution is  $(3, 3)$ .}

4. Consider the weight function  $w(x) = x^{-1/2}$  on the interval  $[0, 1]$ .

- (A) Find the first three monic polynomials

$$p_0(x) = \text{const.}, \quad p_1(x) = x - a, \quad p_2(x) = x^2 + bx + c$$

orthogonal on  $[0, 1]$  with respect to this weight function. In what positions in the complex plane (or real line) can the zeros of  $p_1(x)$  and  $p_2(x)$  be expected to lie? {Do not find the zeros of  $p_2(x)$  explicitly!}

- (B) It is easy to see that if the interval  $[0, 1]$  is replaced by the interval  $[0, h]$  (where  $h > 0$ ) then  $p_1(x)$  is replaced by  $x - ha$ . Use this fact to give a "one-point Gaussian quadrature rule"

$$\int_0^h f(x)x^{-1/2} dx = K \cdot f(A) + \text{error}$$

where  $K$  is a weight and  $A$  is a point in  $[0, h]$  determined using  $p_1(x)$ . Determine  $K$  and  $A$  explicitly (in terms of the parameter  $h$ ). Read up on the small amount of Gaussian quadrature theory needed here in Atkinson, §5.3, p 270 ff., if necessary.

- (C) Derive an error expression for this quadrature formula. The expression should involve a multiple of a value of a suitable-order derivative of  $f(x)$ . {Hint: Try expanding  $f(x)$  in a Taylor series about the point "A".}

5. "Piecewise Linear Splines." Let  $a = x_0 < \dots < x_n = b$  with each  $x_{i+1} - x_i = h_i > 0$ . Let  $f(x)$  be a given  $\mathcal{C}^1$  function on  $[a, b]$ . Let  $s(x)$  be the piecewise linear interpolant of  $f(x)$  which equals the linear interpolant of  $f(x)$  on  $[x_i, x_{i+1}]$  for each  $i = 0, \dots, n - 1$ .

- (A) Show that  $s(x)$  minimizes the integral  $\int_a^b |g'|^2 dx$  among all  $\mathcal{C}^1$  functions  $g(x)$  that interpolate  $f(x)$  at all the  $x_i$ . {Adapt the argument used for the "minimal-energy" characterization of cubic splines to this (simpler!) situation.}

- (B) Give a uniform estimate of the form

$$|f(x) - s(x)| \leq (\text{const.}) \cdot h^c \cdot \left\{ \int_a^b |f'|^2 dx \right\}^{1/2}$$

for a suitable exponent  $c > 0$ , where  $h = \max_i \{h_i\}$ .