642:573:01 MISCELLANEOUS EXERCISES (ASSIGNMENT #4)

1. Let f be a function defined on $[a, b]$ and $a = x_0 < \cdots < x_n = b$ be nodes on the same interval. In each subinterval $[x_{i-1}, x_i]$ introduce two new nodes $x_{i-2/3}$, $x_{i-1/3}$ which are 1/3, 2/3 of the way from left to right respectively. The **Bessel interpolant** of $f(x)$ in this scheme is the piecewise-cubic-polynomial function which is a cubic polynomial on each $[x_{i-1}, x_i]$ and interpolates $f(x)$ on that interval at the four points x_{i-1}, $x_{i-2/3}$, $x_{i-1/3}$ and x_i. Denote the Bessel interpolant of $f(x)$ on $[a, b]$ by $C(x)$.

A. Assuming $f(x)$ has four derivatives, recall that if $x \in [x_{i-1}, x_i]$ then

$$f(x) - C(x) = \frac{1}{4!} f^{(4)}(\xi_i)(x - x_{i-1})(x - x_{i-2/3})(x - x_{i-1/3})(x - x_i)$$

where $x_{i-1} < \xi_i < x_i$, $i = 1, \ldots, n$. Deduce that if the nodes are equally spaced with $x_i - x_{i-1} = h$, then

$$|f(x) - C(x)| \leq \frac{M_4}{4!} \cdot \frac{h^4}{81},$$

where $M_4 = \max\{|f^{(4)}(x)| : a \leq x \leq b\}$, is a bound for the error of approximation that is valid for all $x \in [a, b]$.

B. Apply the result of **A** above to determine what choice of h will be required in order that the Bessel interpolant of the function

$$f(x) = 24e^{3(x-1)}$$

on $[-1, 1]$ approximate it "uniformly on $[-1, 1]$ with error $< 10^{-8}$," *i.e.*, such that for every $x \in [-1, 1]$ the error is bounded by 0.00000 001. Compare this with the number of interpolation points required if the zeros of a Chebyshev polynomial $T_n(x)$ are used to interpolate $f(x)$ in the usual way.

C. Are any of the derivatives of $C(x)$ continuous, in general?

2. Consider the single-step Method (the **simple Kutta method**) given by

$$\begin{aligned} F(x, y, h; f) &= \frac{1}{6}\left[k_1 + 4k_2 + k_3\right] \\ \text{where} \quad k_1 &= f(x, y) \\ k_2 &= f\left(x + \frac{h}{2}, y + \frac{h}{2} k_1\right) \\ k_3 &= f(x + h, y + h \cdot (2k_2 - k_1)) \,. \end{aligned}$$

Note the obvious relation with Simpson's rule of approximate quadrature (more apparent here than in the Runge-Kutta method). **Show** that this is a third-order Method in the sense that

$$\tau(x, Y, h; f) = O(h^3)$$

when $Y(x)$ is a solution of $Y'(x) = f(x, Y(x))$. (Use a symbolic manipulation program—don't make your life harder than it has to be . . . .) Comment on the stability of this Method.

3. Let $[a, b]$ be a real interval. Suppose we have a function E whose domain is the set of closed subintervals of $[a, b]$—so this is a "set function"—and which takes real values. Suppose $e(h)$ is a continuous nonnegative-real-valued function of the positive real variable h which "estimates E" in the sense that

$$|E([c, d])| \leq e(d - c)$$

for each interval $[c, d] \subseteq [a, b]$. Suppose the estimation function satisfies

$$\lim_{h \to 0+} \frac{e(h)}{h} = 0 \quad \text{and} \quad \lim_{h \to +\infty} \frac{e(h)}{h} = \infty \,.$$

(a) Let $\epsilon > 0$ be given, and produce a sequence of points

$$a = x_0 < x_1 < \cdots$$

in the following way: if x_k has been constructed, then x_{k+1} is chosen as $x_k + h_{k+1}$ where h_{k+1} is the largest solution $h_{k+1} > 0$ of

$$e(h_{k+1}) = \frac{\epsilon}{b-a}\, h_{k+1}$$

having the property that if $h < h_{k+1}$ then also

$$e(h) \le \frac{\epsilon}{b-a}\, h$$

unless such an x_{k+1} would be $> b$, in which case one chooses $x_{k+1} = b$. **Show**, using the standard properties of the real numbers, that such an h_{k+1} always exists.

(b) **Show** that the process described in **(a)** always terminates; *i.e.*, at some step *it must happen that* $x_{k+1} = b$. With n denoting the index for which $x_n = b$, **show** that then

$$\sum_{k=0}^{n-1} |E[x_k, x_{k+1}]| \le \epsilon \ .$$

c) The result of **(a)** and **(b)** can be used to provide theoretical justification for adaptive methods of quadrature (for integrals) and adaptive methods for step-size control in solving initial-value problems of o. d. e. For example, in adaptive approximate integration of $f(x)$ over [a,b] using a quadrature method which assigns $I[c, d]$ as the approximate value of the integral $\int_c^d f(x)\, dx$ with an error $E[c, d]$—*e.g.*, for the midpoint method $E[c, d] = (d-c)^3 \frac{f''(\xi)}{24}$—for which an estimate of the type considered in **(a)** and **(b)** is available, one proceeds as follows. Decide to accept a total error of ϵ. Pick x_1 so that applying the method on $[a, x_1]$ incurs an $|\text{error}| \le h_1 \cdot \epsilon/(b-a)$; then pick x_2 so that applying the method on $[x_1, x_2]$ incurs an $|\text{error}| \le h_2 \cdot \epsilon/(b-a), \ldots$. **Explain** how the result of **(a)** and **(b)** guarantees that one will eventually reach $x_n = b$ in finitely many steps, thus theoretically excluding the embarrassing possibility that the adaptive quadrature process will loop forever without ever reaching b. {A similar application, though it's a bit more complicated, applies to adaptive step-size choice in approximate solution of o. d. e. initial-value problems as in the o. d. e. notes.}