Math 574 — Spring 2004

Take-home Final examination

Part A: Twentynine is an interesting number, so let us use it (or, rather, its square root) to illustrate topics discussed in the course.

1. Find the ordinary continued fractions of

$$\alpha = \sqrt{29} \quad \text{and} \quad \beta = \frac{1+\sqrt{29}}{2}.$$

These continued fractions are periodic (after a preperiod).

2. Use the periodic part of the continued fractions of α and β to find the fundamental unit of the quadratic orders of discriminant 29 and $4 \times 29 = 116$.

3. What is the relation between these units?

4. What is a *Gauß-reduced* quadratic form, and what is its relation to continued fractions? Illustrate using the purely periodic parts of the continued fractions of α and β.

5. Find the **five** smallest PV-numbers in the field generated by the square root of 29 over the rationals.

6. In "The inhomogeneous minima of binary quadratic forms, IV", *Acta Math.* **92** (1954), 235–264, E. S. Barnes introduces the notion of an *I-reduced form.* Use his description to list all I-reduced forms of discriminant 29. Identify those that are Gauß-reduced.

7. These I-reduced forms are used to construct *divided cells* for *inhomogeneous diophantine approximations.* Give an example of three consecutive divided cells of an expression, using the list found above.

Part B: Exponentials and logarithms.

1. Sketch a proof of the transcendence of e.

2. What more is needed to prove the transcendence of π by this method?

3. If the goal were only to prove these numbers irrational, can this be done more simply? Explain.

4. Which of the numbers $e+\pi$, $e \times \pi$, e^{π} are known to be transcendental? How is this known?