

Problem Set 1; 640:591, Spring 2009

1. In this problem \mathbb{P} is a finitely additive probability measure.

(a) Prove the inclusion-exclusion formula

$$\mathbb{P}(A_1 \cup \cdots \cup A_n) = \sum_{r=1}^n (-1)^{r+1} \sum_{1 \leq i_1 < \cdots < i_r \leq n} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_r})$$

(b) The event $A_1 \cup \cdots \cup A_n$ may be interpreted as the event that at least one of A_1, \dots, A_n occurs. Let B_m be the event that exactly m of A_1, \dots, A_n occur; B_m is a union of events of the form

$$A_{j_1} \cap \cdots \cap A_{j_m} \cap A_{i_1}^c \cap \cdots \cap A_{i_{n-m}}^c$$

where the indices are all different. Show that

$$\mathbb{P}(B_m) = \sum_{r=m}^n (-1)^{r-m} \binom{r}{m} \sum_{1 \leq i_1 < \cdots < i_r \leq n} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_r})$$

(c) The matching problem is a classical problem of elementary probability theory. N people wearing hats go to a party. When they leave, the hats are re-distributed to the party goers at random, with all re-distributions of hats being equally likely. Find the probability that no one gets back his or her own hat. Find the probability that exactly 1 person gets back his or her own hat.

2. These problems concern elementary facts about algebras and σ -algebras or are simple exercises using the definitions. Do them if you need the review. At least read the problem statements.

(a) Let Ω be a set of infinite cardinality. Let \mathcal{F} be the ensemble of sets A such that either A or its complement is finite. Show that \mathcal{F} is an algebra but not a σ -algebra.

(b) Show that if \mathcal{F} is a σ -algebra (algebra) of subsets of Ω and $g : \Theta \rightarrow \Omega$ then

$$g^{-1}(\mathcal{F}) \triangleq \{g^{-1}(A); A \in \mathcal{F}\}$$

is a σ -algebra (algebra).

(c) The Borel σ -algebra $\mathcal{B}(\mathcal{R})$ is the smallest σ -algebra of subset of \mathcal{R} containing all open sets. Show that the Borel σ -algebra is also generated by (i) all open intervals; (ii) all closed intervals; (iii) all intervals of the form $(a, b]$; (iv) all intervals of the form $(-\infty, b]$.

(d) Let $\{\mathcal{F}_\alpha; \alpha \in \mathcal{A}\}$ be a family of algebras such that for any $\alpha_1, \alpha_2 \in \mathcal{A}$ there is an $\alpha_3 \in \mathcal{A}$ such that $\mathcal{F}_{\alpha_i} \subset \mathcal{F}_{\alpha_3}$ for $i = 1, 2$. Show that

$$\bigcup_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha$$

is an algebra.

(e) Let $\{\mathcal{F}_\alpha; \alpha \in \mathcal{A}\}$ be a family of σ -algebras such that for any countable sequence $\alpha_1, \alpha_2, \dots$ in \mathcal{A} there is an $\beta \in \mathcal{A}$ such that $\mathcal{F}_{\alpha_i} \subset \mathcal{F}_\beta$ for all i . Show that

$$\bigcup_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha$$

is a σ -algebra.

(f) (This is an application of (e)). Let I be an index set, and for each $\alpha \in I$, let \mathcal{F}_α be a σ -algebra of subsets of Ω . Show that for each $A \in \sigma(\bigcup_{\alpha \in I} \mathcal{F}_\alpha)$ there exists a countable subset $\{\alpha_1, \alpha_2, \dots\} \subset I$ such that $A \in \sigma(\bigcup_k \mathcal{F}_{\alpha_k})$.

3. Let $\Omega = \{0, 1\}^\infty$ and let \mathcal{F} denote the σ -algebra generated by the cylinder sets of Ω . Let $0 \leq p \leq 1$. Show that $\{\omega; \lim_{n \rightarrow \infty} n^{-1} \sum_1^n \omega_i = p\}$ is a set in \mathcal{F} .

4. Let \mathcal{R} be an algebra of subsets of Ω , and let \mathbb{P} be a probability measure on $(\Omega, \sigma(\mathcal{R}))$, where $\sigma(\mathcal{R})$ is the σ -algebra generated by \mathcal{R} . Show that for each $A \in \sigma(\mathcal{R})$ and each $\epsilon > 0$, there is a set $A_0 \in \mathcal{R}$ such that $\mathbb{P}(A \Delta A_0) < \epsilon$. ($A \Delta A_0$ denotes the symmetric difference of A and A_0 .) Hint: Show that the collection of events that can be approximated in probability arbitrarily accurately by a set in \mathcal{R} is a σ -algebra.)