Problem Set 2, 640:591, Spring 2009

1. Let $\mathbb{R}^\infty$ be the infinite cross-product of the real line $\mathbb{R}$ and denote a typical element by $\omega = (\omega_1, \omega_2, \ldots)$. Consider also the familiar spaces

$$\ell_2 := \left\{ \omega \in \mathbb{R}^\infty \mid \sum_i \omega_i^2 < \infty \right\}$$

with Hilbertian norm $\|\omega\|_2^2 := \sum_i \omega_i^2 < \infty$. For a positive integer i, the projection operator X_i, given by $X_i(\omega) = \omega_i$, is well-defined on each of these spaces. let A subset of $\mathbb{R}^\infty$ of the form

$$\{\omega \mid (\omega_1, \omega_2, \ldots, \omega_N) \in A\},$$

where A is a Borel subset of $\mathbb{R}^N$ for some positive integer N, is called a cylinder set. Cylinder sets of ℓ_2 are defined similarly The same argument as in Chapter 1 for $\mathbb{R}^\infty$, the cylinder sets of ℓ^2 form an algebra but not a σ-algebra.

a) Let $\mathcal{F}$ denote the σ-algebra of subsets of ℓ_2 generated by the cylinder sets of ℓ_2. It is the smallest σ-algebra that makes every projection operator measurable. Show that $\mathcal{F}$ coincides with the Borel sets of ℓ_2, defined using the topology associated to the norm $\|\cdot\|_2$.

b) Let P be a probability measure on the Borel sets of $I\!R$ such that $P(\{0\}) < 1$. We know from Chapter 1 of the notes how to construct the product measure $P^\infty = P \times P \times \cdots$ on $(I\!R^\infty,\ \otimes_1^\infty \mathcal{B})$, where $\mathcal{B}$ represents the Borel sets of $I\!R$. Recall that P^∞ is the unique extension of the finitely additive measure μ defined on the cylinder sets of $I\!R^\infty$ as follows: if U is the cylinder set

$$\{\omega \in I\!R^\infty \mid (\omega_1, \omega_2, \ldots, \omega_N) \in A\},$$

where A is a Borel subset of $I\!R^N$, then

$$\mu(U) = P^N(A),$$

where P^N is the N-fold product of P. Suppose we try to imitate this construction for ℓ_2 That is, for a cylinder set

$$U = \{\omega \in \ell_2 \mid (\omega_1, \omega_2, \ldots, \omega_N) \in A\},$$

where A is a Borel subset of $I\!R^N$, we define $\nu(U) = P^N(A)$. Show that ν is a finitely additive probability space on the algebra of cylinder sets of ℓ_2 but that it does not admit a countably additive extension.

This example is interesting for the following reason. Let $\{\phi_n\}$ be an orthogonal basis of $L^2([0,1])$ with respect to the usual inner product $\langle f, g\rangle = \int_0^1 fg\,\mathrm{d}x$; for example $\{\sqrt{2}\sin(n\pi x); n \geq 1\}$. Engineers like to model signals in terms of Fourier series.

If a ‘signal’ f is a function in $L^2([0,1])$, it can be represented in terms of its Fourier series

$$\sum_1^\infty \langle f, \phi_n \rangle \phi_n(x)$$

We can model a random signal from $L^2([0,1])$ using a probability measure on ℓ^2 for the sequence of Fourier coefficients $(\langle f, \phi_1 \rangle, \langle f, \phi_2 \rangle, \ldots)$. Essentially, we generate a random signal by choosing the its Fourier coefficients at random. The intuitive idea of white noise is that of a random signal with the same power at each frequency. Mathematically, we interpret this as a model in which the Fourier coefficients are independent and all have the same probability distribution, namely a product measure P^∞ on ℓ^2. The problem above says that we can create a finitely additive probability space on the Borel sets of ℓ^2 corresponding to this idea, but not a countably additive probability space.

2. Suppose that μ and ν are σ-finite measures on a σ-algebra $\mathcal{F}$.

(a) Let $\mathcal{F}_0$ be an algebra which generates $\mathcal{F}$. Suppose that $\mu(A) \le \nu(A)$ for all A in $\mathcal{F}_0$. Show that $\mu(A) \le \nu(A)$ for all $A \in \mathcal{F}$,

(b) Show that (a) can fail if $\mathcal{F}_0$ is only a π-system.

3. Let $\mathcal{C}$ and $\mathcal{G}$ be non-empty collections of subsets of Ω. Recall that a collection $\mathcal{G}$ is called a λ-system if

(i) $\Omega \in \mathcal{G}$;

(ii) $A \in \mathcal{G}$, $B \in \mathcal{G}$, and $B \subset A$ imply that $A - B \in \mathcal{G}$;

(iii) If $A_1, A_2 \ldots$ is an increasing sequence of elements of $\mathcal{G}$, then $\cup_1^\infty A_n \in \mathcal{G}$.

Prove the Dynkin System Theorem: Assume that $\mathcal{C} \subset \mathcal{G}$. If $\mathcal{G}$ is a λ-system and if $\mathcal{C}$ is closed under finite intersections, then $\sigma(\mathcal{C}) \subset \mathcal{G}$.

Hint: Let $\mathcal{D}$ be the smallest λ-system containing $\mathcal{C}$ and prove that $\mathcal{D} = \mathcal{C}$. Note that to do this it suffices to show that $\mathcal{D}$ is closed under finite intersections. Use the type of argument appearing in the proof of the Monotone Class Theorem.

4. Give an example of two collections of sets $\mathcal{G}_1$ and $\mathcal{G}_2$ in a probability space such that $\mathcal{G}_1$ and $\mathcal{G}_2$ are independent, but $\sigma(\mathcal{G}_1)$, and $\mathcal{G}_2$ are not.

5. a) Let A_1, A_2, A_3 be independent events each having probability 1/2. Let B_{ij} be the event that either both A_i and A_j occur or neither occurs. Show that B_{12}, B_{13} and B_{23} are pairwise but not mutually independent.

b) Construct a probability space with $n+1$ events such that any subset of n are mutually independent but all $n+1$ are not independent.