Problem Set 3, 640:591, Spring 2009

1. (A most useful identity.) Let $p \ge 1$. Show that $E[|X|^p] = \int_0^\infty p x^{p-1} I\!P(|X| \ge x)\,\mathrm{d}x$.

2. Let $p \ge 0$. Show that if $E[|X|^p] < \infty$, $x^p I\!P(|X| \ge x) \to 0$ as $x \to \infty$. (Don't try to use problem 1 for this!)

Conversely, show that if $\lim_{x\to\infty} x^p I\!P(|X| \ge x) = 0$, then $E[|X|^{p-\epsilon}] < \infty$, for any $0 < \epsilon < p$.

3. Show that $X_n \stackrel{a.s.}{\to} X$ implies $X_n \stackrel{P}{\to} X$. Give examples for which $X_n \stackrel{P}{\to} X$, but $X_n \stackrel{a.s.}{\to} X$ is not true, for which $X_n \stackrel{a.s.}{\to} X$ but $E[|X_n - X|] \not\to 0$, and for which $E[|X_n - X|] \to 0$, but $X_n \stackrel{a.s.}{\to}$ does not hold.

4. Let $\{X_n\}$ be an increasing sequence of random variables and assume that for some X, $X_n \stackrel{P}{\to} X$. Show that $X_n \stackrel{a.s.}{\to} X$.

5. (a) Let $\{X_n\}$ be a sequence of random variables. The rank Y_n of X_n is

$$Y_n = \sum_{i=1}^{n} \mathbf{1}_{X_i \le X_n}.$$

Assume that $X_1, X_2, \ldots$ are i.i.d. Asssume also that $I\!P(X_i = X_j) = 0$ if $i \ne j$ (for this it suffices that the common distribution of the X_i's contains no atoms). Show that $Y_1, Y_2, \ldots$ are independent and $I\!P(Y_n = k) = 1/n$, $1 \le k \le n$ for each n.

Hints: Let N be the event that there is at least one tie (that $X_i = X_j$ for some i and j). Show that N has probability zero. Hence we can work on the probability space obtained by restricting Ω to $\Omega - N$ without changing the joint distributions of $X_1, X_2, \ldots$ and on this space there are no ties to worry about. For a fixed n, let π be the random permutation of $\{1, \ldots, n\}$ defined by ordering $X_1, \ldots, X_n$ so that $X_{\pi_1} < X_{\pi_2} < \cdots < X_{\pi_n}$. Show that all permutations are equally likely.

(b) A record occurs at n if $X_n > \max\{X_i; i \le n\}$. Show that with probability one there is not a last record.

6. Let X be a random variable and let f and g be bounded increasing functions. Show that $\mathrm{Cov}(f(X), g(X)) \ge 0$.

Hint: This is not obvious from a direct application of the definition of covariance, which says

$$\mathrm{Cov}(f(X), g(X)) = E[(f(X) - \mu_1)(g(X) - \mu_2)] = \int_{R} (f(x) - \mu_1)(g(x) - \mu_2)\,\mathrm{d}F_X(x),$$

where μ_1 and μ_2 are the expected values of $E[f(X)]$ and $E[g(X)]$ respectively. Instead use the independence coupling of two copies of F_X; this is a probability space supporting two independent random variables X' and X'', both of which have the same distribution as X. Show that $\text{Cov}(f(X')-f(X''), g(X')-g(X'')) = 2\text{Cov}(f(X), g(X))$ and use this.

7. Let $X_1, X_2, \ldots$ be i.i.d. with finite second moment. Prove

$$\frac{\max\{|X_i|;\, i \le n\}}{\sqrt{n}} \xrightarrow{P} 0.$$

8. Let $\xi(s) := \sum_1^\infty n^{-s}$ for $s > 1$. Fix s and let X be a random variable taking values in the positive integers such that

$$I\!P(X = n) = n^{-s}/\xi(s) \qquad n \ge 1.$$

(a) Let $D_p := \{p \text{ divides } X\}$. Show that the events $\{D_p | p \text{ is prime}\}$ are mutually independent.

(b) Give a probabilistic proof of Euler's formula

$$\xi^{-1}(s) = \prod_{p \text{ is prime}} (1 - p^{-s}).$$

(c) Show that $I\!P(n^2 \text{ does not divide } X \text{ for all } n > 1) = \xi(2s)$.

(d) Let Y have the same distribution as X and let it be independent of X. Let H be the greatest common factor of X and Y. Find $I\!P(H = n)$.

(Source: D. Williams, *Probability with Martingales*, Cambridge University Press.)

9. Chebyshev's inequality says that $I\!P(|X| \ge a) \le a^{-2}E[X^2]$. Prove the lower bound for X satisfying $E[X] \ge 0$ and $0 \le b < 1$,

$$I\!P(X > bE[X]) \ge (1-b)^2 (E[X])^2 / E[x^2].$$

Hint: Show $(1-b)E[X] \le E[X\mathbf{1}_{\{X > bE[X]\}}]$ and apply Cauchy-Schwarz.

10. Suppose that the moment generateing function $M(t) = E[e^{tX}]$ exists and is finite on $-a < t < a$. Show that M is differentiable on $(-a, a)$ and that $M^{(n)}(0) = E[X^n]$ for every positive integer n, where $M^{(n)}$ denotes the derivative of order n.

11. (If $\{b_n\}$ is a sequence of numbers, when we say that $\lim_{n\to\infty} b_n$ exists, we allow the limit to be infiniite. Let $X_1, X_2, \ldots$ be independent. Show that the event $\{\lim_{n\to\infty} X_n \text{ exists}\}$ has probability one or zero and that if this probability is one, there is an a, $-\infty \le a \le \infty$, such that $\{\lim_{n\to\infty} X_n = a\}$ with probability one.