Problem Set 4, 640:591, Spring 2009

1. By refining the technique used to prove Theorem 3.9 in the notes, show that if $X_1, X_2, \ldots$ are uncorrelated random variables with common mean 0 and common variance $\sigma^2 < \infty$, then

$$\text{(a.s.)}\ \lim \frac{1}{n^{3/4+\delta}} \sum_1^n X_i = 0,$$

for any $\delta > 0$.

2. Let $X_1, X_2, \ldots$ be i.i.d. with exponential distribution: that is, $I\!\!P(X_i > x) = e^{-x}$. Prove that

$$\limsup_{n\to\infty} \frac{1}{\log n} X_n = 1 \qquad \text{almost surely.}$$

3. Let $X_1, X_2, \ldots$ be i.i.d. random variables with $E[X_i] = 0$ and $\gamma = E[X_i^4] < \infty$.
(a) Show that

$$E[(\sum_1^n X_i)^4] \le c\gamma n^2,$$

for some finite constant c independent of γ and n.
(b) Use (a) to prove directly that

$$\text{(a.s.)}\ \lim \frac{1}{n^{3/4+\delta}} \sum_1^n X_i = 0,$$

for any $\delta > 0$.

4. (Source, Shiryayev, **Probability**) Let $X_1, X_2, \ldots$ be a sequence of independent random variables.

(a) Show that if $\sum_1^\infty X_i^2 < \infty$ a.s. then $\sum_1^\infty X_i$ converges a.s. if and only if

$$\sum_1^\infty E[X_i \mathbf{1}_{\{|X_i| \le 1\}}] \qquad \text{converges.}$$

(b) If $\sum_1^\infty X_i$ converges then $\sum_1^\infty X_i^2 < \infty$ a.s. if and only if

$$\sum (E[X_i \mathbf{1}_{\{|X_i| \le 1\}}])^2 < \infty.$$

5. Let $X_1, X_2, \ldots$ be i.i.d. and suppose that $E[X_i] = \infty$ (that is $E[X_i^+] = \infty$ and $E[X_i^-] < \infty$, where X_i^+ and X_i^- are the positive and negative parts of X_i.) Show that $(1/n)\sum_1^n X_i \to \infty$ almost surely.

6. Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with $E[|X_i|] = \infty$. Show that $\limsup_{n\to\infty}(1/n)|\sum_1^n X_i| = \infty$, almost surely. The following preliminary steps may be helpful.

(a) Show that $\mathbb{P}(|X_n| > an \text{ i.o. }) = 1$ for any constant $a > 0$.

(b) Show that $\{ \, |X_n| > 2an \text{ i.o. } \} \subset \{ \, \sum_1^n |X_i| > an \text{ i.o. } \}$.

7. (Source, Williams) When the random variables of a sequence are independent but not identically distributed, funny things can happen apropos the law of large numbers. Let $X_1, X_2, \ldots$ be independent and suppose that $\mathbb{P}(X_n = n^2 - 1) = n^{-2}$ and $\mathbb{P}(X_n = -1) = 1 - n^{-2}$. If we think of X_n as the payoff on play n of a game, we see that we win a lot with small probability and lose a small amount with low probability, but the game seems fair because $E[X_n] = 0$ for each n. However, would we really want to play this game? Show that

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = -1 \qquad \text{a.s.}$$

8. Let $X_1, X_2, \ldots$ be i.i.d. with mean $\mu = 0$ and assume $E[|X_1| \log(1 + |X_1|)] < \infty$. Show that $\sum_1^\infty X_n/n$ converges almost surely. Construct an example to show that the infinite series need not converge if $E[|X_1| \log(1 + |X_1|)] = \infty$.

9. (Problem from Durrett) This problem is a variation on Kolmogorov's Lemma. Let $X_1, X_2, \ldots$ be independent random variables and let $S_{m,n} = \sum_{m+1}^n X_i$. Show that

$$\mathbb{P}(\max_{m<j\le n} |S_{m,j}| > 2a) \max_{m<k\le n} \mathbb{P}(|S_{k,n}| \le a) \le \mathbb{P}(|S_{m,n}| > a).$$

Use this to prove that for the infinite series $\sum_1^\infty X_i$, convergence in probability implies almost sure convergence.

The following exercises develop the Chernoff and Hoeffding bounds. In these exercises,

$$\Lambda_Y(t) = \begin{cases} \ln E[e^{tY}], & \text{if } E[e^{tY}] < \infty; \\ \infty, & \text{otherwise.} \end{cases}$$

and

$$I_Y(a) = \sup_t at - \Lambda_Y(t).$$

10. Let $X_1, X_2, \ldots$ be independent Bernoulli random variables with $\mathbb{P}(X_i = 1) = p$, $\mathbb{P}(X_i = 0) = 1 - p$. Show that $\Lambda_{X_i}(t) = \ln(pe^t + 1 - p)$ and that

$$I_{X_i}(a) = a \ln(\frac{a}{p}) + (1-a) \ln(\frac{1-a}{1-p}),$$

if $0 \leq a \leq 1$ and $I_{X_i}(a) = \infty$ otherwise. Let $S_n = n^{-1} \sum_1^n X_i$ be the empirical mean. Show that if $x \geq p$

$$\mathbb{P}(S_n \geq x) \leq \exp\{-n(x \ln(\frac{x}{p}) + (1-x)\ln(\frac{1-x}{1-p}))\}. \tag{3}$$

11. Let $H(x|p)$ denote the function appearing in the exponent in the bound in (3). Show that $H_{xx}(x|p) \geq 4$ for any p, $0 < p < 1$ and conclude that $H(x|p) \geq 2(x-p)^2$ for $0 < x < 1$. (H_x denotes the partial derivative of H with respect to x.) Derive the Chernoff bound

$$\mathbb{P}(A_n \geq p + \epsilon) \leq e^{-2n\epsilon^2}.$$

This implies the bound

$$\mathbb{P}(A_n \leq p - \epsilon) \leq e^{-2n\epsilon^2}$$

by a symmetry argument.

12.(Hoeffding's inequality) Suppose that Y is a random variable with $a \leq Y \leq b$ almost surely for some finite a and b and assume $E[Y] = 0$. Show that

$$E[e^{tY}] \leq \frac{be^{ta} - ae^{tb}}{b-a}.$$

(Hint: Use convexity of the exponential function.) Use this to show that
$\Lambda_Y(t) \leq (b-a)^2 t^2/8$ and thus that

$$E[e^{tY}] \leq e^{(b-a)^2 t^2/8}. \tag{1}$$

Now suppose that $Y_1, Y_2, \ldots$ are independent, that $E[Y_i] = 0$ for each i, and $a_i \leq Y_i \leq b_i$ almost surely for each i. Use (1) and the general strategy sketched above to show that for $\alpha \geq 0$

$$\mathbb{P}(\sum_1^n Y_i \geq \alpha) \leq \exp\{-2\frac{\alpha^2}{\sum_1^n (b_i - a_i)^2}\}.$$

(Use the bound (1) rather than trying to compute $I_{\sum Y_i}(x)$.)