Solutions, Problem Set 3, Math 591, Spring 2009

Hand in problems 4 and 8.

1. Let $M_X(t)$ be the moment generating function of X, let $\mathcal{D} = \{t;\, M_X(t) < \infty\}$, and $\Lambda_X(t) = \ln M_X(t)$. We know that if there exists t_a such that $\Lambda'_X(t_a) = a$, then the rate function of large deviation theory equals $I(a) = \sup_t(at - \Lambda_X(t)) = at_a - \Lambda_X(t_a)$. Suppose now that there is a $t > 0$ such that $\Lambda_X(t) < \infty$.

(a) Show that $-\infty \le E[X] < \infty$, and $E[X] = \lim_{t\downarrow 0} M'_X(t)$.

(b) We know that $\Lambda_X(t)$ is strictly convex and hence, from (a), that $\Lambda'_X(t) > \lim_{t\downarrow 0} M'_X(t)/M_X(t) = E[X]$ for $a \in \text{int}(\mathcal{D})$. Let $T = \sup\{t;\, M_X(t) < \infty\}$, and suppose $T < \infty$, and $A = \lim_{t\uparrow T} \Lambda'_X(t) < \infty$. If $E[X] < a \le A$, then t_a defined as above exists. Suppose that $A < \infty$. First show that $\Lambda_X(T) < \infty$ and that if $a > A$, $I(a) = aT - \Lambda(T)$. The object of the rest of this problem is to derive a large deviations lower bound by using the change of measure techniques for $X_1, \ldots$ i.i.d. with the distribution of X. As usual, $S_n = \sum_1^n X_i$. By imitating Lemma 6 in the notes on large deviations, show that if $a > A$

$$\mathbb{P}(S_n \ge na) \ge \mathbb{P}(na \le S_n \le a + 3\epsilon) \ge e^{-nT(a+3\epsilon)+\Lambda_X(T)}\mathbb{P}_T(na \le S_n \le a + 3\epsilon),$$

where, under the measure $\mathbb{P}_T$, $X_1, X_2, \ldots$ are i.i.d. with distribution,

$$\mathbb{F}(B) = \frac{E[\mathbf{1}_B(X)e^{TX}]}{M_X(t)} \quad \text{and mean} \quad \frac{E[Xe^{TX}]}{M_X(T)} = A. \tag{1}$$

Now show that

$$\mathbb{P}_T(na \le S_n \le a+3\epsilon) \ge \mathbb{P}_T(n(A-\epsilon) \le S_{n-1} \le n(A+\epsilon))\cdot\mathbb{P}(n(a-A+\epsilon) \le X_n \le (a-A+2\epsilon])$$

and

$$\limsup_{s\to\infty} \frac{1}{s} \ln \mathbb{P}_T(X_1 \in (s, s+1]) = 0.$$

Use these two facts and the weak law of large numbers to deduce that

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mathbb{P}(S_n \ge na) = -I(a).$$

2. Find the characteristic functions of the following distributions. (In all these cases, the characteristic function extends to an analytic function in the complex plane and by letting $\lambda = -it$ in the characteristic function $\phi(\lambda)$ one can recover the moment generating function as well.)

(a) The Bernoulli distribution, $I\!\!P(X=1)=p$ and $I\!\!P(X=0)=1-p$.
(b) The binomial distribution with parameters n and p.
(c) The Poisson distribution, $I\!\!P(X=k)=\frac{\lambda^k}{k!}e^{-\lambda}$ for $k=0,1,2,\ldots$.
(d) The exponential distribution with density $f(x)=\mathbf{1}_{\{x>0\}}\theta e^{-\theta x}$.
(e) The gamma distribution with parameters r and θ given by the density $f(x)=\Gamma^{-1}(r)\theta^r x^{r-1}e^{-\theta x}$.

3. Prove that the sum of independent Poisson random variables is a Poisson random variable. Prove that the sum of independent gamma distributions with the same θ parameter is again a gamma distribution.

4. In each case, determine whether or not the Central Limit Theorem holds for a sequence of independent random variables with the given distributions.
(a) $I\!\!P(X_k=\pm 2^k)=1/2$.
(b) $I\!\!P(X_k=\pm 2^k)=2^{-(1+2k)}, \quad I\!\!P(X_k=0)=1-2^{-2k}$.
(c) $I\!\!P(X_k=\pm k)=1/2\sqrt{k}, \quad I\!\!P(X_k=0)=1-1/\sqrt{k}$.
(d) $I\!\!P(X_k=\pm k^\alpha)=(1/6)k^{-2(\alpha-1)}, \quad I\!\!P(X_k=0)=1-(1/3)k^{-2(\alpha-1)}$.
(e) X_k is uniformly distributed over $(-k,k)$.
(f) $I\!\!P(x_k=\pm k^2)=(1/12)k^{-2}, \quad I\!\!P(X_k=\pm k)=1/12, \quad I\!\!P(X_k=0)=1-(1/6)-(1/6k^2)$. Observe that the Lindeberg condition is not satisfied in this case. Nevertheless,

$$3\sqrt{2}\frac{S_n}{n^{3/2}}$$

converges in distribution to a standard normal random variable as $n\to\infty$. Explain. Hint: Let Y_k be X_k truncated at level k, and show that $\{X_k\}$ and $\{Y_k\}$ are equivalent.

5. Show that if, for some $\delta>0$,

$$\lim_{n\to\infty}\frac{1}{B_n^{2+\delta}}\sum_1^n E[|X_k|^{2+\delta}]=0,$$

then the Lindeberg condition must hold. This is called Lyapunov's condition.

6. Let $X_1,X_2,\ldots$ be independent and suppose X_k has a distribution satisfying

$$F_k(x)=1-(1/2)x^{-2-(1/k)}, \quad \text{for } x>1, \text{ and } \quad F_k(x)=1-F_k(-x), \quad \text{for } x<-1.$$

Show that the Lindeberg condition is not satisfied.

7. The purpose of this problem is to prove the converse of Kolmogorov's three series theorem. Namely, if $X_1,X_2,\ldots$ are independent and if $\sum_1^\infty X_i$ converges, then for every $a>0$, each of the following three series converges

(i) $\sum_1^\infty \mathbb{P}(|X_i| \geq a)$;
(ii) $\sum_1^\infty E[X_i \mathbf{1}_{\{|X_i| \leq a\}}]$;
(iii) $\sum_1^\infty \mathrm{Var}(X_i \mathbf{1}_{\{|X_i| \leq a\}})$.

Let $X_i^a := X_i \mathbf{1}_{\{|X_i| \leq a\}}$. Prove first that series (i) converges. Then prove by contradiction that $\sum_1^\infty \mathrm{Var}(X_i^a) < \infty$, by showing that if $\sum_1^\infty X_i$ converges but $\sum_1^\infty \mathrm{Var}(X_i^a) = \infty$ then

$$\lim_{n\to\infty} B_n^{-1} \sum_1^n X_i^a = 0 \quad \text{and} \quad B_n^{-1} \sum_1^n X_i^a - E[X_i^a] \to \text{a standard normal in distribution,}$$

where $B_n^2 = \sum_1^n \mathrm{Var}(X_i^a)$. Derive from this the contradiction

$$B_n^{-1} \sum_1^n E[X_i^a] \to \text{a standard normal in distribution.}$$

Finally show that condition (ii) holds.

8. (a) Binomial convergence to Poisson. Show that if Y_n is binomial with parameters n and p_n and $np_n \to \lambda > 0$ as $n \to \infty$, then Y_n converges in distribution to a Poisson random variable with parameter λ as $n \to \infty$.

(b) We can represent each Y_n as $X_{n1} + \cdots + X_{nn}$, where $X_{n1}, \ldots, X_{nn}$ are i.i.d. Bernoulli with parameter p_n. Thus the CLT does not hold for this triangular array of random variables. Show analytically why Lindeberg's condition fails.